ELEMENTARY PRACTICAL MATHEMATICS

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A TEXTBOOK COVERING THE SYLLABUSES
OF EXAMINATIONS IN PRACTICAL MATHEMATICS FOR THE
NATIONAL CERTIFICATE

BOOK II

(SECOND YEAR)

BY

E. W. GOLDING
M.Sc.Tech., A.M.I.E.E., Mem.A.I.E.E.
LECTURER IN ELECTRICAL ENGINEERING
UNIVERSITY COLLEGE, NOTTINGHAM

AND

H. G. GREEN
M.A.
READER IN GEOMETRY, UNIVERSITY COLLEGE, NOTTINGHAM

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PREFACE TO THE SECOND EDITION

Since the book was first published, several slight modifications to the text have been made, and a considerable number of examples added as the opportunity arose at times of reprinting.

In this edition, several more extensive alterations to the text have been made, the main object of which has been to introduce certain useful alternative methods of treatment. Experience in using the book has shown also the desirability of some rearrangement of the sets of examples. This has been carried out and extra examples have been added, so that the total number is now almost double that of the original edition.

It has been the authors' endeavour to make the book as useful as possible to second-year students, and it is their sincere belief that these modifications will contribute materially to that end.

E. W. GOLDING
H. G. GREEN

PREFACE TO THE FIRST EDITION

In preparing this textbook of Practical Mathematics, designed primarily to meet the needs of students working for the Ordinary National Certificate, the authors have aimed at providing a tool for direct application to elementary practical problems. Their experience with Engineering students working this stage of the course leads them to the belief that a mathematically rigorous presentation of every process is actually a hindrance to progress, and reduces the students' power to apply the subject usefully. They have therefore adopted the course of establishing the methods, as far as possible, by means of illustrative examples. These examples have been chosen carefully to incorporate the principles of rigorous proof without formal presentation. In addition, almost all the examples are drawn from engineering practice.

The complete work has been divided into three books, one for each year of a three years' course. In so doing, some degree of latitude has had, of necessity, to be observed on account of

the divergence between the syllabuses of the various bodies examining for the National Certificate. Book I covers the first year's work very fully; so that, if thought desirable, such sections as are not required for the first grade examination may be omitted. Book II covers the second year, and Book III the third.

It will be noticed that the Differential Calculus has been carried to a more advanced stage in Book II, and the Integral Calculus in Book III, than is generally required in the examinations. The authors feel that it is strongly desirable that the methods of these subjects should be available at early stages, even if they are not studied for examination purposes, as they are frequently needed at the beginning of the succeeding year's study in technical subjects.

The chapters have been divided into sections by means of short sets of examples, and, at the conclusion of each chapter, there are miscellaneous examples which include many questions from past public examination papers. The authors hope that this arrangement will prove helpful to teachers in planning the weekly work. Revision sets of examples on the work of the preceding books are given at the commencements of Books II and III.

For generously granted permission to use questions from past examination papers they have to thank the following examination bodies: The East Midland Educational Union; The Northern Counties Technical Examination Council; The Union of Educational Institutions; The Union of Lancashire and Cheshire Institutes. The sources of questions thus used are indicated in brackets after the examples.

The authors' thanks are due to their colleague, L. E. Prior, Esq., M.Sc., and to J. H. Green, Esq., M.A., late Principal, Holmfirth Secondary School and Technical Institute, for help in their reading of the proofs; their acknowledgments are also due to the Syndics of the University Press, Cambridge, for permission to reproduce mathematical tables from Godfrey and Siddons's *Four-figure Tables*.

To Sir Isaac Pitman & Sons, Ltd., the authors are indebted for continuous help in the arrangement and presentation of the books.

E. W. GOLDING
H. G. GREEN

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EXAMPLES ON REVISION OF BOOK I

(1) Find in miles per hour the average speed of an aeroplane which flew six times round a triangular course, sides 97·8 miles, 123·9 miles and 117·1 miles, in 5 hr. 23 min. (*E.M.E.U.*)

(2) The value of π lies between 3·14159 and 3·1416. If the approximate value $\frac{22}{7}$ is used for π when calculating the circumference of a circle of diameter 280 yd. show that the resulting error lies between 12·67 and 12·78 in. (*N.C.*)

(3) A piece of mild steel rectangular in cross-section was tested in a testing machine by pulling it till fracture occurred. The cross-sectional dimensions originally were 2·014 in. by 0·387 in., and after fracture these were 1·524 in. by 0·250 in. Express the reduction in cross-sectional area as a percentage of the original cross-sectional area. (*U.E.I.*)

(4) (*a*) If $V = \dfrac{\pi h}{3}(A + \sqrt{AB} + B)$ find h when $V = 456$, $A = 27$, $B = 12$.

(*b*) The areas A, B, C of three fields satisfy the equations $9A = 7B$; $5B = 6C$. If the total area of the three fields is 94 acres find A, B and C. (*E.M.E.U.*)

(5) Write down each of the following and fill in the blank spaces—

(i) $(2{\cdot}3 \times 5{\cdot}8) + (2{\cdot}3 \times 17{\cdot}9) = 2{\cdot}3\,(\qquad\qquad)$;

(ii) $\dfrac{x}{x+y} = \dfrac{2x}{\qquad}$;

(iii) $$\begin{aligned} a^3 + b^3 &= (a^3 + a^2b) - (a^2b + ab^2) + (ab^2 + \qquad) \\ &= a^2(\qquad\qquad) - ab(\qquad\qquad) + b^2(\qquad\qquad). \\ &= (a+b)(\qquad\qquad\qquad). \end{aligned}$$

Evaluate with as little labour as possible

$$\frac{(5{\cdot}9)^3 + (4{\cdot}1)^3}{(5{\cdot}9)^2 - (5{\cdot}9)(4{\cdot}1) + (4{\cdot}1)^2}.$$ (*N.C.*)

(6) The heating effect of an electric current may be calculated from the following formula—

$$H = \frac{I^2Rt}{4{\cdot}2}.$$

Find the value of I when $H = 25{,}000$, $R = 110$, $t = 300$.

State the effect on the value of H of

(i) halving the value of R,
(ii) doubling the value of I. (*U.L.C.I.*)

(7) (*a*) Find the value of E from the formula—

$$E = 1{\cdot}434\,\{1 - 0{\cdot}00077\,(t - 15)\},\ \text{when } t = 30.$$

(*b*) Find m from the following equation—

$$N = \frac{m}{2(m+1)}E,\ \text{given } N = 12,\ E = 30.$$ (*U.L.C.I.*)

(8) (*a*) In an experiment where steam was passed into water the following equation was used to calculate t, a required temperature in degrees Fahrenheit—

$$0{\cdot}3L + 0{\cdot}3\,(212 - t) = 6(t - 60).$$

Calculate t to the nearest degree when $L = 970$.

(*b*) Using logarithms find the value of $23 \times 7{\cdot}14 \div 0{\cdot}479$. (*U.E.I.*)

(9) (*a*) Define the logarithm of a number to a given base.

(*b*) Find by logarithms the value of

(i) $\dfrac{81{\cdot}06 \times 0{\cdot}0634}{3{\cdot}74 \times 0{\cdot}7843}$; (ii) $\sqrt[3]{15{\cdot}36}$. (*E.M.E.U.*)

(10) (*a*) Express each of the numbers 5, 15, 25, 250 as a power of 10.

(*b*) Evaluate $1000^{0{\cdot}4}$, $\sqrt[5]{1000}$.

(*c*) By means of logarithms evaluate

$$\frac{3{\cdot}41 \times \sqrt{341}}{\sqrt[3]{341^2}}.$$ (*N.C.*)

(11) Solve—

(*a*) $\dfrac{V-(7-V)}{3} = 11$,

(*b*) $2{\cdot}832t = 49{\cdot}17$;

(*c*) $15{\cdot}5 = \dfrac{23{\cdot}4}{s} + 20{\cdot}18$ (*E.M.E.U.*)

(12) Find the value of—

(*a*) $\dfrac{13{\cdot}6 \times 737 \times 273}{760 \times 288}$;

(*b*) $\sqrt{73{\cdot}4} \times (0{\cdot}831)^2$. (*E.M.E.U.*)

(13) By means of the slide rule adjust the numbers 74, 56, 35, 27 and 23 so that the highest is represented by 250.

Sketch the settings of the rule when dealing with the number 27. (*U.E.I.*)

(14) Using the mathematical tables supplied evaluate the following expressions—

(*a*) $\sqrt{\dfrac{3S(L-S)}{8}}$, when $S = 260$, $L = 260{\cdot}027$;

(*b*) $K \tan \theta$, when $K = 1{\cdot}5$, $\theta = 48°$. (*U.L.C.I.*)

(15) (*a*) Fill in the spaces in the following table—

Angle	Sine	Cosine	Tangent
26°	. . .	.	. . . .
	0·6018		. . . .
	.	0·6293	. .
. . . . .	. .	. .	0·1405

(*b*) A ladder 20 ft. long rests against a wall with its lower end on the ground. The foot of the ladder is 8·45 ft. from the wall. Find the angle the ladder makes with the ground. (*U.E.I.*)

(16) (*a*) Draw an angle A such that $\sin A = \frac{3}{4}$, and measure the angle. Using measurements from the same figure find $\cos A$ and $\tan A$, and check these values by means of the tables.

(*b*) If $\sin A = \frac{3}{4}$, find the value of $\dfrac{1 - \tan^2 A}{1 + \tan^2 A}$. (*E.M.E.U.*)

(17) The length of a rectangular block is three times its height, and its breadth is half its length; its height is h in. Obtain a formula for its volume in terms of h.

If the volume of the block is 36 cub. in. what is its height? (*N.C.*)

(18) The base of a pyramid is a square of side 6 ft., and the perpendicular height is 4 ft. Find (*a*) the volume of the pyramid, (*b*) the area of one of its slant faces. (*N.C.*)

(19) The x and y co-ordinates of three points A, B and C are given in the following table—

Point	A	B	C
x (inches)	2	-3	-3
y (inches)	1	-2	2

Draw the triangle ABC to a scale of full size and find its area in square inches. (*U.L.C.I.*)

(20) A solid bar of steel is in the shape of a cylinder with a conical end. The cylindrical portion has a diameter of $\frac{1}{2}$ in. and is 6 in. long. If the total length of the bar is 7 in. find the volume of the bar. (Take $\pi = 3{\cdot}14$, and give your answer correct to three significant figures.) (*N.C.*)

(21) (*a*) If $\dfrac{1}{a} - \dfrac{2}{b} = \dfrac{3}{b} - \dfrac{4}{c}$ find c in terms of a and b.

Find the value of c when $a = 2$, $b = 4$.

(*b*) The floor of a room is a rectangle a ft. long and b ft. wide. How many planks x ft. long and y in. wide will be required to cover it if an area of c sq. ft. is reserved for the fireplace? (*E.M.E.U.*)

(22) A metal pipe has an external diameter of 4 in. and an internal diameter of $3\frac{3}{4}$ in. Find (*a*) the cross-sectional area of the metal of the pipe, (*b*) the weight of a length of 8 ft. if the metal weighs $\frac{1}{4}$ pound per cubic inch. (Take $\pi = 3{\cdot}14$ and use logarithms to calculate your answer.) (*N.C.*)

(23) A cast-iron weight should be 5 lb., but weighs 4·98 lb. To correct this a hole $\frac{3}{8}$ in. diameter is drilled in the weight and then plugged with lead. Calculate in inches to three significant figures how deep the hole should be. The weights of 1 cub. in. of cast iron and of lead are 0·26 lb. and 0·41 lb. respectively. (*U.E.I.*)

(24) The co-ordinates of a point A are $x = 2$, $y = 1$, and of a point B, $x = 6$, $y = 4$. Find the length of AB and the angle (to the nearest degree) which a straight line through A and B makes with the axis of x. (*U.L.C.I.*)

(25) The following table gives related values of pressure (P) and volume (V) during the expansion of a gas—

P	110	55	36·6	27·5	22	18·3	15·7	13·7	12·2	11	10
V	2	4	6	8	10	12	14	16	18	20	22

Plot the curve connecting P and V, and find the average value of P during the expansion. *(U.L.C.I.)*

(26) Taking 1 in. for one unit on each axis, draw the rectangle bounded by the straight lines

$$y = 3,\ y = -5,\ x = 1,\ x = -4.$$

Find the equation of each of its diagonals. *(E.M.E.U.)*

(27) In the following table is given a series of Centigrade thermometer readings (C.) and the corresponding Fahrenheit readings (F.)

C.	− 10	30	50	70	90	120
F.	14	86	122	158	194	248

Plot to as big a scale as the paper will allow F vertically and C horizontally Assuming that the connection between F and C may be expressed in an equation of the form $F = a\,C + b$ where a and b are constants, find from the graph the value of the constants a and b *(U.E.I.)*

ELEMENTARY PRACTICAL MATHEMATICS

CHAPTER I

ARITHMETIC AND ALGEBRA

(Sections and questions which may, if desired, be left for a second reading are marked with an asterisk.)

Arithmetic

Proportion. By far the greatest number of arithmetical processes in practical life are dependent on the simple idea of *proportion*. Such statements as "a car goes 35 miles to the gallon; it will therefore need 6 gallons for 210 miles," or "I had 10 minutes to catch my train; I have gone half way to the station in 5 minutes so I may just catch it" are nothing but elementary applications of the idea of proportion, and in the discussion of formulae in Chapter II, Book I, we have already made extensive use of it. In arithmetic many of the questions which are usually classified under rules with complicated titles can be dealt with easily and simply by a common-sense application of this idea, and we give some examples worked out on these lines, though the proportion is not always formally stated. The critical step is almost always of one of two types, of which the following statements are representative.

1. Eight men do a piece of work in 6 days. Then 1 man will do it in 48 days or 48 men in 1 day; the 48 (called *man-days*) is a measure of the work.

2. It takes me 4 days to do a piece of work, then I do $\frac{1}{4}$ of it in one day.

The use of these steps is to reduce various sections of the question to a common basis.

Example 1. Forty men can do a piece of work in 21 days; if 5 men fall out at the end of every 6 days how long will it take them?

Number of man-days required $= 40 \times 21 = 840$.

During the	1st 6 days	40 men do	240 man-days				
,, ,,	2nd ,,	35 ,,	210 ,, ,,	;	total done	= 450	man-days
,, ,,	3rd ,,	30 ,,	180 ,, ,,	;	,, ,,	= 630	,, ,,
,, ,,	4th ,,	25 ,,	150 ,, ,,	;	,, ,,	= 780	,, ,,

There now remain 60 man-days, which will be completed by the remaining 20 men in 3 days, or in all $4 \times 6 + 3 = 27$ days will be spent on the work.

EXAMPLE 2. A man and a boy do a piece of work in 6 days. The man alone can do it in 8 days. In how many days can the boy alone do it if, when alone, he only works at $\frac{3}{4}$ the rate at which he works when with the man? (*E.M.E.U.*)

In one day the man and the boy together do $\frac{1}{6}$ of the work
" " " the man alone does $\frac{1}{8}$ " "
$\therefore$ " " " the boy (working with the man) " $\frac{1}{6} - \frac{1}{8} = \frac{1}{24}$ of the work.
$\therefore$ " " " the boy working by himself " $\frac{3}{4} \times \frac{1}{24} = \frac{1}{32}$ of the work

or by himself it would take him 32 days.

EXAMPLE 3. Twenty-four men working 7 hr. a day dig a trench 105 ft. by 8 ft. by 6 ft. in 5 days. How many hours a day will 16 men have to work to dig a trench 64 ft. by 12 ft. by 3 ft. in 4 days?

In the first case volume $= 105 \times 8 \times 6$ cub. ft., and man-hours required $= 24 \times 7 \times 5$.

In the second case volume $= 64 \times 12 \times 3$ cub. ft., and man-hours required is therefore

$$\left(\frac{64 \times 12 \times 3}{105 \times 8 \times 6}\right) \times (24 \times 7 \times 5).$$

But 16 men are available for 4 days.

$$\therefore \text{ hours per day} = \frac{64 \times 12 \times 3}{105 \times 8 \times 6} \times \frac{24 \times 7 \times 5}{16 \times 4}$$

which on cancelling reduces to 6.

Note. In questions involving intermediate steps as in this one, arithmetical reductions should be left as long as possible.

EXAMPLE 4. A well, full of water, into which the influx is 10 gal. per min. is pumped dry in 30 min. by a pump drawing 100 gal. per min. If a second pump is installed it is pumped dry in 15 min. What is the output of the second pump?

With the first pump the well loses 90 gal. per min. and is emptied in 30 min.

$\therefore$ to be emptied in 15 min. it must lose $\dfrac{90 \times 30}{15} = 180$ gal. per min.

$\therefore$ the output of the second pump must be $180 - 90 = 90$ gal. per min.

EXAMPLE 5. A stream fills a well in 270 min. With the

stream still running a pump empties the well in 30 min. How long would it have taken two pumps of the same power to empty it?

In 1 min. the stream fills $\frac{1}{270}$ of the well.

In 1 min. with the pump and stream, $\frac{1}{30}$ of the well is emptied.

$\therefore$ In 1 min. the pump alone empties $\frac{1}{30} + \frac{1}{270}$ of the well.

$\therefore$ In 1 min. with two pumps and the stream

$2\left(\frac{1}{30} + \frac{1}{270}\right) - \frac{1}{270} = \frac{19}{270}$ of the well is emptied, or

the well will be emptied in $\frac{270}{19} = 14\frac{4}{19}$ min.

Note. Compare closely with Example 4.

EXAMPLE 6. Two wheels, geared together, have 80 and 70 teeth. The first revolves at 245 revolutions per minute; at what rate does the second revolve?

For one revolution of the first wheel 80 teeth of the second will have been engaged, or the second will have made $\frac{80}{70}$ revolutions. Hence its number of revolutions per minute will be

$$\frac{80}{70} \times 245 = 280.$$

Note. If we have a gear train the rate of the final wheel will be independent of the numbers of teeth on the intermediate wheels; thus, for example, if a train consists of wheels having 80, 100, 120, 95, 70 teeth, and the first is turning at 245 rev. per min., the last will turn at

$$\frac{80}{100} \times \frac{100}{120} \times \frac{120}{95} \times \frac{95}{70} \times 245$$

$$= \frac{80}{70} \times 245 \text{ cancelling the 100's, 120's, 95's}$$

$$= 280 \text{ r.p.m.}$$

EXAMPLES I

(1) Eight men can do a piece of work in 6 days; after 2 days 3 men fall out; after 3 more days 2 more men fall out. How long will it take the rest to finish the work?

(2) One man can do a piece of work in 4 days and a second can do it in 6 days. How long will it take them to do it working together?

(3) A and B do a piece of work in 6 days; A alone could do it in 10 days. In what time could B do it alone?

(4) One man does a piece of work in 12 days but two men together, working more efficiently, can do it in 5 days. Two men and a boy do it in 4 days If the men's rate of working is not affected by the presence of the boy how long would it take a man and a boy to do it?

(5) A trench 60 ft. by 9 ft. by 4 ft. is excavated by 5 men in 3 days How long would it take 8 men to excavate a trench 120 ft. by 8 ft. by 6ft. working the same number of hours a day?

(6) A cistern can be filled by two inlet taps in 18 min. When it is half full one tap is shut off and the second completes the filling in 12 min. Find how long it would take the first tap to fill it from empty.

(7) Two men run at uniform rates. One runs 100 yd. in $10\frac{1}{5}$ sec. and beats the other by $2\frac{1}{2}$ yd. By how long will he beat him?

(8) A cargo boat sailing from a port X calls in turn at Y and Z and then returns direct to X; the distances are X to Y 100 miles, Y to Z 150 miles, Z to X 200 miles. For one merchant it carries 150 tons from X to Y and 36 from X to Z; for another it carries 80 tons from Y to Z and 140 from Y to X. If their total bill is £170 what should each pay?

Ratios. If the weight of one machine is 20 tons and that of a second machine is 36 tons we may say that their weights are in the ratio of 20 to 36 (usually written 20: 36); we might equally have said that the weight of the first $= \frac{20}{36}$ that of the second, and it is then clear that a ratio obeys the same rule for cancelling as a fraction, that is

$$20 : 36 = 5 : 9.$$

It is important to notice that we can only have a ratio between quantities of the same type; we *cannot* have a ratio between 12 pigs and 8 horses, but we *can* have the ratio "the *number* of pigs to the *number* of horses is 12 : 8" (the ratio is then between two numbers). Further, the two quantities must not only be of the same general type, but they must also be measured in the same units. Thus, to find the ratio between weights of 4 lb. and 12 oz. we have

either $\quad 4 \text{ lb.} = 4 \times 16 \text{ oz.}$

and the ratio $= (4 \times 16) : 12 = 16 : 3$ cancelling through by 4,

or $\quad 12 \text{ oz.} = \frac{12}{16} \text{ lb.}$

and the ratio $= 4 : \frac{12}{16} = 4 : \frac{3}{4} = 16 : 3$ multiplying through by 4.

The idea of a ratio can be immediately extended from two to any number of quantities of *the same* type; thus, if for four pieces of metal, the weights of which are given as

	A	B	C	D
	4 lb.	12 oz.	18 oz.	24 oz.
i.e.	64 oz.	12 oz.	18 oz.	24 oz.

we may say that the weights are in the ratio

$$
\begin{array}{rcccccccc}
 & 64 & : & 12 & : & 18 & : & 24 \\
= & 32 & : & 6 & : & 9 & : & 12.
\end{array}
$$

It is important to note that to obtain the second line we have divided right through by 2; we may divide (or multiply) by anything we like, but we must do it to *every* number of the ratio sequence.

The methods of using ratios are given in the following worked-out examples, which the student should read carefully. He will notice the frequency with which the "idea" of parts is introduced.

EXAMPLE 1. Divide a weight of 48 lb. into two parts in the ratio of $5:7$.

If the weight were divided into 12 (i.e. $5 + 7$) equal sections one part would consist of 5 of them and the other of 7, or the parts are

$$5 \times \tfrac{48}{12} = 20 \text{ lb., and } 7 \times \tfrac{48}{12} = 28 \text{ lb.}$$

EXAMPLE 2. The outputs of machines A and B are in the ratio $5:6$; the outputs of machines C and B are in the ratio $3:4$. Find the ratios of the outputs of A and C.

We have

$$
\begin{array}{lcc}
 & A:B & B:C \\
 & 5:6 & 4:3 \\
\text{or} & (5 \times 4):(6 \times 4) & (4 \times 6):(3 \times 6).
\end{array}
$$

Note the inversion; the question gives C to B $3:4$.

5×4 parts for A and 3×6 parts for C now both correspond to the same number of parts for B; namely 24 ($= 6 \times 4$ or 4×6), or the outputs for A and C are in the ratio

$$(5 \times 4):(3 \times 6) = 10:9, \text{ cancelling through by 2.}$$

EXAMPLE 3. Two points of an electric circuit are connected by three wires; the current in each wire is inversely proportional to the resistance. If the resistances are in the ratios $4:6:8$, and the current in the third wire is $2\frac{1}{2}$ amperes, find the total current.

$$
\begin{array}{ll}
\text{Ratios of resistances} & 4:6:8 \\
\therefore \text{ ratios of currents} & = \frac{1}{4}:\frac{1}{6}:\frac{1}{8} \\
 & = 6:4:3 \text{ multiplying through by 24}
\end{array}
$$

If these numbers represent parts then

3 parts are $2\frac{1}{2}$ amperes

and the total current equals $6 + 4 + 3 = 13$ parts

and is $2\frac{1}{2} \times \frac{13}{3} = 10\frac{5}{6}$ amperes.

EXAMPLE 4. The weights of two cubes, one of wood and one of iron, are in the ratio 7 : 9. The volumes of equal weights of wood and iron are in the ratio 19 : 2. Find the ratio between the volumes of the cubes.

$$\frac{\text{The volume of 1 lb. of wood}}{\text{the volume of 1 lb. of iron}} = \frac{19}{2}$$

$$\therefore \quad \frac{\text{the volume of 7 lb. of wood}}{\text{the volume of 9 lb. of iron}} = \frac{19 \times 7}{2 \times 9} = \frac{133}{18}.$$

But the weights of the cubes are in the same ratio as 7 lb. to 9 lb.

$\therefore$ their volumes are in the ratio 133 : 18.

EXAMPLE 5. The weights of two cubes, one of wood and one of iron, are in the ratio 7 : 9. The weights of equal volumes are in the ratio 2 : 19. Find the ratio between the volumes of the cubes.

$$\frac{\text{The weight of 1 cub. ft. of wood}}{\text{the weight of 1 cub. ft. of iron}} = \frac{2}{19}$$

$$\therefore \quad \frac{\text{the weight of } \frac{1}{2} \text{ cub. ft. of wood}}{\text{the weight of } \frac{1}{19} \text{ cub. ft. of iron}} = \frac{1}{1}$$

$$\therefore \quad \frac{\text{the weight of } \frac{7}{2} \text{ cub. ft. of wood}}{\text{the weight of } \frac{9}{19} \text{ cub. ft. of iron}} = \frac{7}{9}$$

$\therefore$ the volumes of the cubes are in the ratio

$$\frac{7}{2} : \frac{9}{19} = 7 \times 19 : 9 \times 2 = 133 : 18.$$

Note. Compare the wordings of Examples 4 and 5 very closely.

EXAMPLE 6. In one pile of fuel the ratio by weight of coal to coke is 2 : 3 and in a second pile it is 9 : 2. If the piles are mixed, the weights of coal and coke are in the ratio 2 : 1. What is the ratio of the total weights of the piles?

Suppose the ratio is $x : 1$.

In the first pile $\frac{2}{5}$ is coal and $\frac{3}{5}$ is coke.

In the second pile $\frac{9}{11}$ is coal and $\frac{2}{11}$ is coke.

$\therefore$ for x tons of the first pile and 1 ton of the second, since the total weight of coal is twice that of coke,

$$\frac{2}{5}x + \frac{9}{11} = 2\left(\frac{3}{5}x + \frac{2}{11}\right)$$

or

$$\frac{4x}{5} = \frac{5}{11}$$

$$x = \frac{25}{44}$$

and the required ratio is 25 : 44.

EXAMPLES II

(1) Divide 72 into two parts in the ratio of 13 : 5.

(2) A number is divided into three parts in the ratios of 7 : 9 : 21. The smallest of the parts is 35; find the number.

(3) Two taps A and B fill a tank in 1 hr. 25 min. If the ratio of their flows is 5 : 6, how long will it take A to fill the tank alone?

(4) Three streams A, B, C supply a well. If the flows of A and B are in the ratio 7 : 2, and those of B and C in the ratio 5 : 6, what is the ratio of the flows of A and C?

If the total flow is 28,500 gal. per min. what is the flow of each?

(5) The heights of two objects of the same material and shape are in the ratio of 7 : 2. The ratio between their weights equals the ratio between the cubes of their heights. If the smaller one weighs 24 lb. what is the weight of the heavier?

(6) The areas of two objects of the same shape are in the ratio of the squares of corresponding lengths. On one map the area occupied by Great Britain is 8·45 sq. cm. and on another it is 2·45 sq. cm. What is the ratio between the scales on which the maps are drawn?

(7) One alloy of metal A and metal B contains them in the ratio 5 : 4, and a second contains them in the ratio 7 : 5. Equal quantities of each are mixed. In what ratio are the alloys present in the mixture?

(8) An alloy of metal A and metal B contains them in the ratio of 5 : 4, and a second alloy contains them in the ratio of 7 : 3. By melting the two together a third alloy is obtained containing A and B in the ratio of 12 : 7. If 50 tons of the second alloy were used what was the total weight of the mixture?

ALGEBRA

Linear Simultaneous Equations. We found in Chapter VII, Book I, that the graph of an equation of the first degree in x and y is a straight line, and that, if we are given two such equations, the x and y co-ordinates of the point of intersection

of the two lines which form their graphs are the values of x and y which satisfy both the equations. Two equations of this type are called *linear simultaneous equations*, and the process of finding the values of x and y is called *finding their solution*.

Algebraic Solution. We are given two equations in x and y. To solve these equations we obtain from the two a single equation in x (or y) only and then proceed by the ordinary methods for simple equations. The process will be most clearly explained by means of some worked-out examples.

(The formation of an equation containing x only is conveniently termed *eliminating y*.)

EXAMPLE 1. Solve the equations
$$4x + 3y = 8 \qquad \text{(i)}$$
$$5x + 7y = 4 \qquad \text{(ii)}$$

We notice that $3y$ occurs in the first equation and $7y$ in the second; in order that we may have the same number of y's in the two equations, we multiply the first equation right through by 7 and the second by 3. This does not affect the equalities and we obtain,

multiplying (i) by 7,

$$28x + 21y = 56$$

multiplying (ii) by 3,

$$15x + 21y = 12.$$

Then the differences of the equals, $(28x + 21y) - (15x + 21y)$ and $56 - 12$, will be equal, so that, by subtraction,

$$13x = 44$$

$$\therefore \qquad x = \frac{44}{13}.$$

To find y,

from (i) $\qquad 3y = 8 - 4x$

$$= 8 - 4 \times \frac{44}{13} = 8 - \frac{176}{13} = -\frac{72}{13}$$

or $\qquad y = -\frac{24}{13}$

and the solution of the equations is $x = \frac{44}{13}, y = -\frac{24}{13}$.

We have in this solution first obtained x and then, using the equation (i), have obtained y from it; we can check the accuracy of our work by testing in equation (ii)

$$5x + 7y = 4.$$

When $x = \frac{44}{13}$ and $y = -\frac{24}{13}$, $5x + 7y = \frac{220 - 168}{13} = \frac{52}{13} = 4.$

Note 1. The solution of the equations should always be given in the form $x = \frac{44}{13}$, $y = -\frac{24}{13}$, *not* Ans. $= \frac{44}{13}, -\frac{24}{13}$, as this does not state clearly which is x and which is y. For the sake of brevity, however, the answers printed at the end of the book will be given in this form (with the x value first); but it should not be used by the student.

Note 2. The correctness of answers should always be checked when practicable. We show the check in the early examples, but have omitted it in the later ones to save space.

Example 2. Solve the equations

$$5x + 6y = 8 \qquad \text{(i)}$$
$$2x - 4y = 5 \qquad \text{(ii)}$$

We note that 12 ($= 6 \times 2$ or 4×3) is the smallest number containing 6 and 4.

Multiplying (i) by 2 $\quad 10x + 12y = 16$

multiplying (ii) by 3 $\quad 6x - 12y = 15$

adding $\quad 16x = 31$

$$\therefore \qquad x = \frac{31}{16}$$

from (ii) $\quad 4y = 2x - 5$

$$= 2 \times \frac{31}{16} - 5 = \frac{31}{8} - 5 = -\frac{9}{8}$$

or $\quad y = -\frac{9}{32}$

and the solution of the equations is $x = \frac{31}{16}$, $y = -\frac{9}{32}$.

Check. Using equation (i), $5x + 6y = 5 \times \frac{31}{16} - 6 \times \frac{9}{32}$

$$= \frac{155}{16} - \frac{27}{16} = \frac{128}{16} = 8.$$

Example 3. Solve the equations

$$x - 3y + 4 = 0 \qquad \text{(i)}$$
$$5x = 2y - 7 \qquad \text{(ii)}$$

We first rewrite the equations in the standard form

from (i) $x - 3y = -4$ (iii)

from (ii) $5x - 2y = -7$ (iv)

multiplying (iii) by 5 $5x - 15y = -20$

subtracting from (iv) $13y = 13$

or $y = 1$

from (i) $\therefore$ $x - 3 + 4 = 0$ or $x = -1$

and the solution of the equations is $x = -1$, $y = 1$.

Check. Using equation (ii), $5x = -5$, $2y - 7 = 2 - 7 = -5$, and the solution is correct.

EXAMPLE 4. Solve the equations $\frac{1}{3}(x-1) - y = x - \frac{1}{5}(y-1) = 2x - 3y$.

Separating the equalities, we have

$$\tfrac{1}{3}(x-1) - y = 2x - 3y \qquad \text{(i)}$$

$$x - \tfrac{1}{5}(y-1) = 2x - 3y \qquad \text{(ii)}$$

from equation (i) $x - 1 - 3y = 6x - 9y$

or $-5x + 6y = 1$ (iii)

from equation (ii) $5x - y + 1 = 10x - 15y$

or $-5x + 14y = -1$ (iv)

subtracting (iv) from (iii) $-8y = 2$

or $y = -\frac{1}{4}$

from (iii) $5x = 6y - 1 = -\frac{3}{2} - 1 = -\frac{5}{2}$

or $x = -\frac{1}{2}$

and the solution of the equations is $x = -\frac{1}{2}$, $y = -\frac{1}{4}$.

Check. As we have not used the equations in their original form in the actual solution, we must check in all the expressions.

When $x = -\frac{1}{2}$, and $y = -\frac{1}{4}$

$$\tfrac{1}{3}(x-1) - y = \tfrac{1}{3}(-\tfrac{3}{2}) + \tfrac{1}{4} = -\tfrac{1}{4},$$

$$x - \tfrac{1}{5}(y-1) = -\tfrac{1}{2} - \tfrac{1}{5}(-\tfrac{5}{4}) = -\tfrac{1}{4},$$

$$2x - 3y = -1 + \tfrac{3}{4} = -\tfrac{1}{4},$$

and the solution is correct.

Note. In Examples 1, 2, 3 one of the original equations was used to give one answer directly, and it was sufficient to check in the other equation; in

Example 4 we have to check throughout in order to guard against slips in the first stage when we rearrange the equations.

* *Literal Equations.* In our previous examples we had equations which connected the unknowns by means of numbers; in literal equations some or all of these numbers are replaced by general symbols, a, b, . . ., but the principle of the solution is exactly the same as before.

EXAMPLE 5. Solve the equations

$$ax - by = ab \tag{i}$$

$$\frac{x}{a} + \frac{y}{b} = \frac{b}{a} - \frac{a}{b} \tag{ii}$$

Multiplying equation (ii) by b^2

$$\frac{b^2x}{a} + by = \frac{b^3}{a} - ab$$

adding to (i) $\left(a + \frac{b^2}{a}\right)x = \frac{b^3}{a}$

or $$x = \frac{b^3}{a^2 + b^2}$$

from (i) $$by = ax - ab$$

$$= \frac{ab^3}{a^2 + b^2} - ab = -\frac{a^3b}{a^2 + b^2}$$

or $$y = -\frac{a^3}{a^2 + b^2}$$

and the solution of the equations is

$$x = \frac{b^3}{a^2 + b^2}, \; y = -\frac{a^3}{a^2 + b^2}.$$

Check. Using equation (ii)

$$\frac{x}{a} + \frac{y}{b} = \left(\frac{b^3}{a} - \frac{a^3}{b}\right) \Big/ (a^2 + b^2) = \frac{b^4 - a^4}{ab} \Big/ (a^2 + b^2)$$

$$= \frac{b^2 - a^2}{ab} = \frac{b}{a} - \frac{a}{b},$$

and the solution is correct.

EXAMPLES III

Solve the equations

(1) $2x + 3y = 8,\ 5x + 2y = 7.$
(2) $4x + 5y = 4,\ x - y = 1.$
(3) $x - 2y = 3,\ 2x - y = 5.$
(4) $3x - 2y - 4 = 0,\ 8x - 7y + 1 = 0.$
(5) $x - 4y + 6 = 0,\ 3x - 2y + 2 = 0.$
(6) $2x - y = 3,\ 7x + 10y = 6.$
(7) $x - 3y = 8,\ 3x - 2y = 3.$
(8) $2x - y + 1 = 0,\ 5x - 7y + 16 = 0.$
(9) $x + y + 1 = 0,\ 11x - 3y + 3 = 0.$
(10) $3x - 4y + 3 = 0,\ 2x + 9y = 5.$
*(11) $ax - by = a + b,\ x + y = 1.$
*(12) $a(ax + b) = b(by + a),\ a(bx - a) = b(ay - b).$

**Miscellaneous Simultaneous Equations.*

EXAMPLE 6. Solve the equations

$$2x + \frac{1}{y+1} = 3 \qquad \text{(i)}$$

$$x - \frac{2}{y+1} = 4 \qquad \text{(ii)}$$

We take, temporarily, x and $\frac{1}{y+1}$ as our unknowns.

Multiplying equation (ii) by 2

$$2x - \frac{4}{y+1} = 8$$

subtracting from (i) $$\frac{5}{y+1} = -5$$

or $$\frac{1}{y+1} = -1$$

and $$1 + (y + 1) = 0$$

$\therefore$ $$y = -2$$

from (ii) $$x - 2(-1) = 4$$

or $$x = 2,$$

and the solution of the equations is $x = 2,\ y = -2$.

Check. In (i) $2 \times 2 + \frac{1}{-2+1} = 4 - 1 = 3$ and the solution is correct.

EXAMPLE 7. Solve the equations

$$2^x + 3^y = 5 \qquad \text{(i)}$$

$$2^{x+1} + 4 \times 3^y = 18 \qquad \text{(ii)}$$

We re-write equation (ii) as

$$2 \times 2^x + 4 \times 3^y = 18$$

or $$2^x + 2 \times 3^y = 9 \qquad \text{(iii)}$$

and, taking 2^x and 3^y as the unknowns, solve equations (i) and (iii) to obtain

$$2^x = 1 \quad \therefore\ x = 0$$

and $$3^y = 4 \quad \therefore\ y \log 3 = \log 4$$

$$y = \frac{\cdot 6021}{\cdot 4771} = 1{\cdot}26$$

and the solution is $x = 0$, $y = 1{\cdot}26$.

EXAMPLE 8. Solve the equations

$$x^2 + y = 4 \qquad \text{(i)}$$

$$2x^2 + 3y = 3 \qquad \text{(ii)}$$

These are not linear equations in x and y, but we can consider them as linear equations in x^2 and y and, solving in the ordinary way, obtain $x^2 = 9$, $y = -5$. If $x^2 = 9$ then x is either 3 or -3, and the solutions of the equations are

$$x = 3,\ y = -5$$

or $$x = -3,\ y = -5.$$

In Examples 8 and 9 the check should be made in equations (ii).

EXAMPLE 9. Solve the equations

$$2x + y + 3z = 13 \qquad \text{(i)}$$

$$x - y + z = 2 \qquad \text{(ii)}$$

$$3x + 5y - 2z = 7 \qquad \text{(iii)}$$

In this case we have three unknowns x, y and z connected by three equations. From two of the equations we eliminate one of the unknowns, and from another pair of equations we eliminate the same unknown. We then have two equations between two of the unknowns, which we solve as before.

adding equations (i) and (ii) $\quad 3x + 4z = 15$

adding 5 times (ii) to (iii) $\quad 8x + 3z = 17$

solving these two equations we obtain $x = 1$, $z = 3$

from (ii) we then have $\quad 1 - y + 3 = 2$

$$y = 2$$

and the solution of the equations is $x = 1$, $y = 2$, $z = 3$.

Check. From (i) $2x + y + 3z = 2 + 2 + 9 = 13$

(iii) $3x + 5y - 2z = 3 + 10 - 6 = 7$

and the solution is correct.

Note. The method of Example 9 can clearly be extended to any number of unknowns provided we have exactly the same number of equations as we have unknowns.

Problems Leading to Linear Simultaneous Equations. In Chapter III, Book I, we gave examples of the solution of problems in which the information given in the question was represented by an equation of the first degree in one unknown. It is often, however, more convenient (and sometimes almost necessary) to use two or more unknowns, and we now give some examples of this type.

EXAMPLE 1. A boy buys a number of articles at 1s. each and a number at 1s. 4d. each, spending in all £1 12s. If he had bought three times as many at 1s. and half as many at 1s. 4d. he would have bought 7 more. How many did he buy of each kind?

Suppose he buys x articles at 1s. each and y at 1s. 4d. each. From the information on total cost, working in shillings, we have

$$x + \frac{4y}{3} = 32 \qquad \text{(i)}$$

and from the information on total numbers

$$\left(3x + \frac{y}{2}\right) - (x + y) = 7 \qquad \text{(ii)}$$

From equation (i) $\quad 3x + 4y = 96 \qquad$ (iii)

from equation (ii) $\quad 4x - y = 14 \qquad$ (iv)

multiplying (iv) by 4 $\quad 16x - 4y = 56$

adding to (iii) $\quad 19x = 152$

or $\quad x = 8$

and from (iv) $\quad y = 4 \times 8 - 14 = 18.$

He bought 8 at 1s. each and 18 at 1s. 4d. each.

Check. Total cost = 8s. + 24s. = 32s.

Numbers, $8 + 18 = 26$, $3 \times 8 + \frac{1}{2} \times 18 = 33$, or 7 more than 26.

Note. When checking problems it is always advisable to go back to the question itself, not to the equations.

EXAMPLE 2. A factory contains machines of two designs. If one machine of type A and two of type B are working their total output per day is 420 lb. One machine of each type is started to carry out an order of 3660 lb.; after two days the type B machine breaks down and after two further days three machines of the same type are started, and the order is completed five days later. Find the daily output of each type of machine.

Let x lb. be the daily output of type A and y lb. that of type B.

From the total daily output $x + 2y = 420$ (i)

In the 9 days the output of the type A machine is $9x$, and the total output of the type B machines is $2y + 5\,(3y) = 17y$.

From the total output $\therefore\ 9x + 17y = 3660$ (ii)

multiplying (i) by 9 $\quad 9x + 18y = 3780$

subtracting (ii) $\quad y = 120$

and from (i) $\quad x = 420 - 2 \times 120 = 180.$

The daily outputs are 180 lb. for type A and 120 lb. for type B.

Check. $180 + 2 \times 120 = 180 + 240 = 420$

$9 \times 180 + 2 \times 120 + 5 \times 3 \times 120 = (162 + 24 + 180)10$
$= 3660$, and the solution is correct.

EXAMPLE 3. The power to drive a factory is derived from three sources; if the first source is trebled and the third cut off the power supplied is 330 h.p.; if the second is doubled the h.p. is 310; if the first is cut off the h.p. is 140. Find the h.p. drawn from each source.

Let x, y, z h.p. be the powers supplied.

First case $\quad 3x + y = 330$ (i)

second case $\quad x + 2y + z = 310$ (ii)

third case $\quad y + z = 140$ (iii)

Subtracting (iii) from (ii) $\quad x + y = 170$

subtracting from (i) $\quad 2x = 160$

or $\quad x = 80$

from (i) $\quad y = 330 - 3 \times 80 = 90$

from (iii) $\quad z = 140 - 90 = 50.$

The powers are 80, 90, and 50 h.p.

EXAMPLES IV

In Questions 1 to 4 solve the equations

*(1) $\frac{2}{x}-\frac{3}{y}=6$, $\frac{3}{x}-\frac{2}{y}-5=0$.

*(2) $4^x+3^y=11$, $3\times 4^x+4\times 3^y=42$.

*(3) $x^3+y^3=1$, $2x^3+3y^3=11$.

*(4) $2x+3y+z=15$, $x+y-z=7$, $3x-y-2z=4$.

(5) A rod is cut into two sections: three times one section exceeds the other by 17 in. and seven times the first with five times the second is 58 in. Find the lengths of the sections.

(6) Two men, starting at the same time from points 8 miles apart and walking towards each other at steady rates, meet after 1 hr. If the first had walked at 2 m.p.h. less than double his speed and the second at 1 m.p.h. more than half his speed they would still have met after 1 hr. What were their speeds?

(7) The total cost for labour and materials to a builder carrying out apiece of work is £480. If the cost for labour increases by 10 per cent, and the cost for materials decreases by 5 per cent, the total cost becomes £495. Find the original costs for labour and materials.

(8) The perimeter of a rectangular plate is 160 in. If the length were 3 per cent longer and the breadth 5 per cent longer, the perimeter would be 166 in. Find the length and breadth

(9) A man bought two fields for £300 and sold them for £337. If he made 10 per cent profit on one and 15 per cent on the other (both on the cost prices), how much each did they cost him?

(10) A man bought three fields A, B and C for £360 A and B cost him equal sums, but he sold them at profits of 12 per cent and 10 per cent respectively, and he sold C at a profit of 15 per cent (all on cost price). If his total profit was £46, how much did each field costhim?

(11) The sum of the digits in a number of two digits is 16. If the digits are interchanged, the number is decreased by 18. Find the number.

(12) The sum of the digits in a number of two digits is 11. If the tens digit is doubled and the units digit is halved, the number is increased by 26. Find the number.

(13) Using 16 machines of one type and 11 of another, each making the same article, a manufacturer's output is 528 articles per day. If he replaces all his machines of the second type by machines of the first, his output is increased by $12\frac{1}{2}$ per cent. How many articles are made per day on each type of machine?

(14) Using 9 machines of one type and 7 of another, each making the same article, a manufacturer's output is 287 articles per day. If he uses 7 of the first type and 9 of the second, the output is 305 a day. How many are made per day on each type of machine?

Quadratic Equations. An equation in one unknown containing powers up to the square is called a *quadratic equation.* Quadratic equations with more than one unknown are usually beyond our present scope, but we give the simplest cases at the end of the chapter.

Many problems in practical work reduce to quadratic equations, and their solution is of the greatest importance. We have already, in Chapter VII, Book I, given one method of obtaining a graphical solution, and further graphical work on them will

be given in Chapter V. We must here consider the algebraic solution.

Method 1: *by Factors*

EXAMPLE 1. Solve the equation $x^2 - 5x + 6 = 0$.

Now $x^2 - 5x + 6 = (x - 3)(x - 2)$, and the equation can therefore be re-written

$$(x - 3)(x - 2) = 0,$$

and it is clear that $x^2 - 5x + 6$ will be zero if, and only if, one of the factors $(x - 3)$ or $(x - 2)$ is zero, that is, if x equals 3 or 2; the solution of the equation is $x =$ either 3 or 2.

Note. The 3 and 2 are usually referred to as the *roots* of the equation, and the question may be worded "Find the roots of the equation . . ."

EXAMPLE 2. Solve the equation

$$2x^2 + 9x = 5.$$

Re-writing in standard form

$$2x^2 + 9x - 5 = 0$$

factorizing $\quad (2x - 1)(x + 5) = 0$

$\therefore \quad x = \frac{1}{2}$ or -5.

EXAMPLE 3. Solve the equation

$$px^2 + qx - p + q = 0.$$

Multiplying through by p

$$(px)^2 + q(px) - p(p - q) = 0$$

factorizing $\quad (px + p)(px - p + q) = 0$

$\therefore \quad px = -p$ or $p - q$

$$x = -1 \text{ or } (p - q)/p.$$

Note. We can multiply (or divide) through by any number or symbol which does not contain the unknown; we *cannot* multiply or divide by the unknown or anything containing the unknown.

The process of solution by factors can be given in the form of the rule: "*Write the quadratic equation as an expression equal to zero, and factorize the expression. Each value of x which makes a factor zero is a root of the equation.*"

Some further examples of solution by factors are given in tabular form.

	Equation	*Factor form*	*Roots*
(i)	$6x^2 + x - 1 = 0$	$(3x - 1)(2x + 1) = 0$	$\frac{1}{3}, -\frac{1}{2}$.
(ii)	$2x^2 + 8x + 6 = 0$	$2(x + 1)(x + 3) = 0$	$-1, -3$.
(iii)	$4x^2 - 8x = 0$	$4x(x - 2) = 0$	$0, 2$.
(iv)	$x^2 + 2x + 1 = 0$	$(x + 1)^2 = 0$	$-1, -1$.

Note. In (ii) the 2 in the factor form does not affect the roots.
(iii) We *cannot* say $4x^2 - 8x = 0$, therefore dividing through by $4x$, $x - 2 = 0$ and therefore $x = 2$. (We can divide through by the 4 if we wish.)
(iv) The roots are equal.

Method 2: by "Completion of the Square." In this method we must keep in mind the formulae

$$a^2 + 2ab + b^2 = (a + b)^2,$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

(In the illustrative examples the answers are given to two decimal places.)

EXAMPLE 1. Solve the equation $x^2 + 5x + 3 = 0$.

We first re-write the equation as $x^2 + 2\left(\frac{5}{2}\right)x = -3$.

Comparing the left-hand side with the left-hand side of the first identity we see that we have x for a and $\frac{5}{2}$ for b, and adding $\left(\frac{5}{2}\right)^2$ (i.e. b^2) to both sides to complete the similarity

$$x^2 + 2\left(\frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2 = -3 + \frac{25}{4}$$

or

$$\left(x + \frac{5}{2}\right)^2 = \frac{13}{4}$$

$\therefore$ taking the square roots of both sides

$$\left(x + \frac{5}{2}\right) = \text{either } \frac{\sqrt{13}}{2} \text{ or } -\frac{\sqrt{13}}{2}$$

and the solution is therefore

$$x = \text{either } -\frac{5}{2} + \frac{\sqrt{13}}{2} = -0{\cdot}70$$

or

$$-\frac{5}{2} - \frac{\sqrt{13}}{2} = -4{\cdot}30.$$

Note 1. When the $\left(\frac{5}{2}\right)^2$ is added to the left-hand side great care must be taken to add the correct equivalent $\frac{25}{4}$ to the right-hand side.

Note 2. Either $\frac{\sqrt{13}}{2}$ or $-\frac{\sqrt{13}}{2}$ is usually written $\pm\frac{\sqrt{13}}{2}$ (plus or minus root thirteen over 2—notice also that on taking the square root it is over 2, not 4), and the answer is written as $\frac{-5 \pm \sqrt{13}}{2}$.

EXAMPLE 2. Solve the equation

$$3x^2 - 5x - 4 = 0.$$

Dividing by 3 $\qquad x^2 - \frac{5}{3}x - \frac{4}{3} = 0$

re-writing $\qquad x^2 - 2\left(\frac{5}{6}\right)x = \frac{4}{3}$

completing the square

$$x^2 - 2\left(\frac{5}{6}\right)x + \left(\frac{5}{6}\right)^2 = \frac{4}{3} + \frac{25}{36}$$

or $\qquad \left(x - \frac{5}{6}\right)^2 = \frac{73}{36}$

$$x - \frac{5}{6} = \pm \frac{\sqrt{73}}{6}$$

or $\qquad x = \frac{5 \pm \sqrt{73}}{6} = 2{\cdot}26 \text{ or } -0{\cdot}59.$

EXAMPLES V

In Questions 1 *to* 9 *solve the equations by the method of factors.*

(1) $x^2 - 7x + 12 = 0.$ (2) $x^2 - 5x - 24 = 0.$ (3) $x^2 + 12x + 27 = 0.$
(4) $2x^2 - 5x + 3 = 0.$ (5) $6x^2 + 5x = 4.$ (6) $8x^2 + 6x + 1 = 0.$
(7) $9x^2 - 27x = 0.$ (8) $4x^2 + 1 = 4x.$ (9) $25x^2 - 10x + 1 = 0.$

In Questions 10 *to* 18 *solve the equations by the method of completing the square. Where the answers are approximate give them to two decimal places.*

(10) $x^2 - 7x + 2 = 0.$ (11) $x^2 - 3 = 5x.$ (12) $2x^2 - 9x + 2 = 0.$
(13) $3x^2 + 9x - 5 = 0.$ (14) $4x^2 + 1 = 8x.$ (15) $2x^2 + 7x + 2 = 0.$
(16) $4x^2 - 13x + 9 = 0.$ (17) $6x^2 - 23x + 20 = 0.$ (18) $12x^2 - 55x + 28 = 0.$

Method 3: *by Formula.* We take as the standard quadratic expression

$$ax^2 + bx + c = 0,$$

and obtain the formula for its roots by the method of completing the square.

$$ax^2 + bx + c = 0$$

$$\therefore \qquad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\therefore \qquad x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$$

$$\therefore \qquad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

or
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \qquad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

and the formula for the roots is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It will be seen at once that the formula is really a convenient summary of the process of completing the square. In its use great care must be taken that the signs are correct, and until the student is fully master of it he should tabulate the values of a, b, c as shown in brackets in the illustrative examples. The method by factors should only be used when the factors can be seen at sight.

Points which are a source of frequent errors in the formula, and which must be specially noticed are

(1) the 2 of $2a$
(2) the 4 of $-4ac$
(3) the $-b$ is over the $2a$

EXAMPLE 1. Solve the equation $8x^2 - 21x + 10 = 0$.

$$\left(a = 8, b = -21, c = 10, \; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$x = \frac{+21 \pm \sqrt{(-21)^2 - 4 . 8 . 10}}{2 . 8}$$

$$= \frac{+21 \pm \sqrt{441 - 320}}{16} = \frac{+21 \pm \sqrt{121}}{16}$$

$$= \frac{+21 \pm 11}{16} = \frac{32}{16} \text{ or } \frac{10}{16}$$

$$= 2 \text{ or } \frac{5}{8}.$$

Note. b is -21, $-b$ is then $+21$, $b^2 = (-21) \times (-21) = +441$.

EXAMPLE 2. Solve the equation $3x^2 + 4x - 6 = 0$.

$$\left(a = 3, b = 4, c = -6, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \,.\, 3 \,.\, (-6)}}{2 \,.\, 3}$$

$$= \frac{-4 \pm \sqrt{16 + 72}}{6} = \frac{-4 \pm \sqrt{88}}{6}.$$

$$= \frac{-4 \pm 9{\cdot}381}{6} = \frac{5{\cdot}381}{6} \text{ or } -\frac{13{\cdot}381}{6}$$

$$= 0{\cdot}90 \text{ or } -2{\cdot}23 \text{ to two decimal places.}$$

Note. In writing down $4ac$ as $4 \,.\, 3 \,.\, (-6)$ we use the bracket to avoid the danger of confusion in sign.

EXAMPLE 3. Solve the equation $x^2 - x - 7 = 0$.

$$\left(a = 1, b = -1, c = -7, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$x = \frac{+1 \pm \sqrt{(-1)^2 - 4 \,.\, 1 \,.\, (-7)}}{2 \,.\, 1} = \frac{+1 \pm \sqrt{29}}{2}$$

$$= 3{\cdot}19 \text{ or } -2{\cdot}19 \text{ to two decimal places.}$$

EXAMPLE 4. Solve the equation $2x^2 + 8x + 5 = 0$.

$$\left(a = 2, b = 8, c = 5, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \,.\, 2 \,.\, 5}}{2 \,.\, 2} = \frac{-8 \pm \sqrt{24}}{4}$$

$$= -0{\cdot}78 \text{ or } -3{\cdot}22 \text{ to two decimal places.}$$

EXAMPLES VI

Solve by formula the equations (giving approximate answers to two decimal places)—

(1) $2x^2 - 5x + 1 = 0$. (2) $5x^2 + 7x - 8 = 0$. (3) $3x^2 - 5x - 8 = 0$.
(4) $x^2 + 13x + 3 = 0$. (5) $7x^2 - 23x + 5 = 0$. (6) $2x^2 - 3x - 6 = 0$.
(7) $8x^2 - 54x + 63 = 0$. (8) $24x^2 + 22x - 35 = 0$. (9) $30x^2 + 61x + 30 = 0$.
(10) $x^2 - 0{\cdot}3x - 3 = 0$. (11) $0{\cdot}4x^2 + 0{\cdot}16x - 1 = 0$. (12) $-3x^2 + 7x + 2 = 0$.
(13) $24x^2 - 74x + 35 = 0$. (14) $27x^2 - 51x - 28 = 0$.
(15) $12x^2 + 25x + 12 = 0$. (16) $x^2 - 0{\cdot}07x - 0{\cdot}045 = 0$.
(17) $5x^2 + 30x + 44{\cdot}2 = 0$. (18) $-0{\cdot}7x^2 + 0{\cdot}53x + 0{\cdot}6 = 0$.

Problems Leading to Quadratic Equations. We now give some examples of problems in which the equation representing the information given in the question is quadratic in one unknown. The quadratic equation gives rise to two roots, and it frequently happens that one of these roots does not correspond to a *practical* solution of the problem. An inquiry into the practical applicability of the roots (and rejection where necessary) is an *essential part of the answer.*

EXAMPLE 1. The perimeter of a rectangular lawn is 336 ft. and its area is 7040 sq. ft. Find the lengths of its sides.

Let x ft. be the length of one side; the length of half the perimeter is 168 ft. and the length of another side is therefore $(168 - x)$ ft. Hence from the area

$$x(168 - x) = 7040$$

$$\therefore \quad x^2 - 168x + 7040 = 0$$

$$x = \frac{168 \pm \sqrt{168^2 - 4 \, . \, 7040}}{2}$$

which gives $x = 80$ or 88

$\therefore$ for the other side $\quad 168 - x = 88$ or 80.

Both roots are applicable, but lead to the same solution—the lawn measures 80 ft. by 88 ft.

EXAMPLE 2. The price of goods is £1 per gross with a reduction of as much per cent as the number of gross available. Find how many gross there were in a parcel of goods sold for £22 15s.

If there were x gross the price per gross would be £$\left(1 - \frac{x}{100}\right)$,

$$\therefore \quad x\left(1 - \frac{x}{100}\right) = 22\tfrac{3}{4}$$

or $$x^2 - 100x + 2275 = 0$$

$$x = \frac{100 \pm \sqrt{100^2 - 4 \, . \, 2275}}{2}$$

which gives $\quad x = 35$ or 65

and both answers are clearly possible.

(We might interpret the 35 as being below the normal demand of the market and the 65 as representing a glut.)

EXAMPLE 3. Equal squares are cut from the corners of a piece of rectangular sheet metal 21 in. by 16 in. so that the area of each square is equal to the area of the central rectangle left in the sheet. Find the length of the side of the square.

Let x in. be the side of the square (see Fig. 1), then the sides

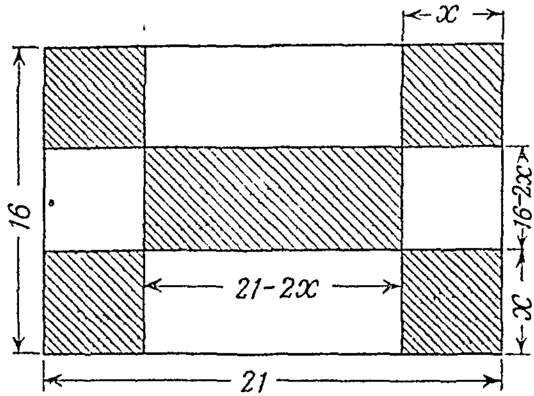


FIG. 1

of the central rectangle are $21 - 2x$ and $16 - 2x$, and from the equality of areas

$$(21 - 2x)(16 - 2x) = x^2$$

or $$3x^2 - 74x + 336 = 0$$

$\therefore$ $$x = \frac{74 \pm \sqrt{74^2 - 4 . 3 . 336}}{6}$$

which gives $x = 6$ or $18\frac{2}{3}$.

If $x = 6$ the rectangle left is 9 by 4.

If $x = 18\frac{2}{3}$ the sides of the rectangle are negative.

Hence the only practical result is $x = 6$ in.

EXAMPLE 4. I buy 75 tons of goods of two sorts. One sort costs £76, and the other, at £2 per ton more, costs £88. What is the cost per ton of the first sort?

Suppose the first sort costs £x per ton, then I buy $\frac{76}{x}$ tons of it, and $\frac{88}{x+2}$ tons of the second sort, and $\frac{76}{x} + \frac{88}{x+2} = 75.$

Multiplying through by $x\,(x+2)$

we have $\quad 76x + 152 + 88x = 75x^2 + 150x$

or $\quad 75x^2 - 14x - 152 = 0$

$$x = \frac{14 \pm \sqrt{14^2 + 4\,.\,75\,.\,152}}{150},$$

which gives $\quad x = -\frac{4}{3}$ or $\frac{38}{25}$.

A negative result is clearly inadmissible, and the answer is £$\frac{38}{25}$ or £1 10s. 4⅘d. per ton.

EXAMPLES VII

(1) The product of two numbers is 294; their sum is 35. Find the numbers.

(2) Of the two sides of a right-angled triangle which include the right angle the length of one exceeds that of the other by 7 ft., and the area of the triangle is 30 sq. ft. Find the length of the shorter side.

(3) The hypotenuse of a right angled triangle exceeds one side by 1 in. and the other by 8 in. Find the length of the hypotenuse.

(4) When a number of men are working together the percentage increase in the output per man is equal to the number of men employed. What number of men working together is equivalent to 24 men working separately?

(5) Two sheets of metal are of equal area. One is twice as long as it is broad. The breadth and length of the second exceeds the breadth of the first by 8 in. and 12 in. respectively. Find the breadth of the first.

(6) Two men walk 21 miles. If one goes 1 mile an hour faster than the other, and takes 1 hr. 20 min. less time, how fast does he go?

(7) The length of a rectangular plate exceeds the breadth by 14 in. and the area is 792 sq. in. Find the breadth.

(8) The perimeter of a rectangular field is 286 yd. and its area is 1 ac. Find the lengths of the sides.

(9) A man spends £10 10s. on one quality of cloth and £10 on a cheaper quality, of which he buys 4 more yards. If the difference in price is 2s. per yard, how many yards of the best quality cloth did he buy?

(10) In a 400-mile race, one car gives another 34 miles start and travels 8 miles per hour faster. If the faster car reaches the finishing post 5 min. before the other, at what rate (in m.p.h.) did it travel?

(11) The area of an isosceles triangle is 60 sq. cm. and the equal sides are 13 cm. long. Calculate the height. [*Note*—If x is the height, the equation for x is quadratic in x^2: there are two solutions.]

(12) The area of a rectangular plate is 12 sq. in. and the perimeter is 14/5 of a diagonal. Calculate the lengths of the sides.

***Simultaneous Equations: One Quadratic, One Linear.** In equations of this type we use the linear equation to eliminate one of the unknowns from the quadratic equation.

Example 1. Solve the equations

$$y^2 + xy + 2x - 22 = 0 \qquad \text{(i)}$$

$$3x + 2y = 11 \qquad \text{(ii)}$$

In equation (i) x enters to the first power and y to the second. It will therefore be easier to eliminate x than y.

From (ii) $$3x = 11 - 2y$$

multiplying (i) by 3 to avoid fractions when we replace x

$$3y^2 + 3xy + 6x - 66 = 0$$

substituting for $3x$, $$3y^2 + (11 - 2y)y + 2(11 - 2y) - 66 = 0$$

$$\therefore \quad y^2 + 7y - 44 = 0$$

$$\therefore \quad (y - 4)(y + 11) = 0$$

or $$y = 4 \text{ or } -11.$$

If $y = 4$, $3x = 11 - 2y = 3$ $\quad \therefore x = 1$.

If $y = -11$, $3x = 11 - 2y = 33$ $\quad \therefore x = 11$,

and we have for the answer—

either $$x = 1 \quad y = 4,$$

or $$x = 11 \quad y = -11.$$

Note. In writing down the answer care must be taken to pair the values of x and y correctly. An answer in which the pairing is not clear or is incorrect is erroneous—thus for example the answer must *not* be given as $x = 1$ or 11, $y = 4$ or -11, as the pairing is not shown.

EXAMPLE 2. Solve the equations $3x^2 + 7y^2 = 10$ (i)

$5x - 2y = 7$ (ii)

In this case x and y are to the second power in equation (i); as y has the smaller coefficient in equation (ii) it will be easier to substitute for y.

From (ii) $$2y = 5x - 7$$

multiplying (i) by 4 ($= 2^2$) to avoid fractions when we replace y

$$12x^2 + 7 \times 4y^2 = 40$$

substituting for $2y$, $12x^2 + 7(5x - 7)^2 = 40$

which gives $$x = 1 \text{ or } \frac{303}{187}.$$

If $x = 1$, $\quad 2y = 5x - 7 = -2$ $\quad \therefore y = -1$.

If $x = \frac{303}{187}$, $\quad 2y = 5x - 7 = \frac{206}{187}$ $\quad \therefore y = \frac{103}{187}$,

and we have for the answer—

either $$x = 1,\ y = -1$$

or $$x = \frac{303}{187},\ y = \frac{103}{187}.$$

(Additional elementary examples on the work of this chapter will be found on page 177.)

EXAMPLES VIII

(1) A man and a boy together do a piece of work in 4 days; the man alone can do half of it in 3 days. How long would it take the boy to finish it, their rates of working being unchanged?

(2) Using 8 machines a piece of work is done in 12 working days of 5 hours each. To what must the length of the working day be changed in order that 20 machines can perform 4 times the work in 16 days?

(3) A mixture of 3 parts sand to 2 of cement is mixed with five times the quantity of a mixture of 4 parts sand to 3 of cement. How many parts of sand to cement are there in the final mixture?

(4) A bicycle is geared to 80 (this means it goes forward 80 in. for each revolution of the pedal); a second bicycle is geared to 90. If their speeds are in the ratio 12 : 13 what is the ratio between the rates of revolution of the pedals.

(5) Four squares are cut out of sheet metal; the weights of the squares are in the ratios $9 : 16 : 12\frac{1}{4} : 25$, and the sum of the perimeters of the four squares is 31 in. What is the perimeter of the smallest?

(6) Solve the equations
$$3x + 8y - 5 = 0,$$
$$2x - 3y + 6 = 0.$$

*(7) Solve the equations
$$5^x + 3^y = 28,$$
$$5^{x+1} + 3^{y+1} = 134.$$

(8) The base of a column stands partly on sand and partly on gravel, and its area is 8 sq ft. If 3 sq. ft. are on the sand the column will bear a load of 97,000 lb.; if 3 sq. ft. are on the gravel the bearing load is 71,000 lb. What are the bearing loads per square foot of sand and gravel?

(9) Solve the equation $2x^2 - 13x = 250$, giving the answers to three significant figures.

(10) Solve the equation $\frac{7}{x+5} - \frac{2}{x-3} = 3$.

(11) A rectangular lawn is surrounded by a path of uniform width. The outside measurements of the path are 25 ft. by 16 ft. Find to the nearest inch the width of the path if its area equals that of the lawn.

*(12) Solve the equations
$$x^2 + xy = 6,$$
$$x + 3y = 5.$$

*(13) Solve the equations
$$x^2 + xy + y^2 = 3,$$
$$3x - 2y = 1.$$

(14) If in the equation $s = ut - \frac{1}{2}at^2$, $s = 240$ when $t = 7\cdot5$, and $s = 7\cdot5$ when $t = 2\cdot5$, find u and a. Hence express s in terms of t. *(E.M.E.U.)*

(15) Solve the equations

(1) $\frac{x}{3} + 5 = \frac{2y}{3},\ y - x = \frac{x}{3}.$

(2) $\frac{x^2}{12} - x + \frac{5}{3} = 0.$ *(E.M.E.U.)*

(16) Find the value of a in order that 2 may be a root of the equation $x + \frac{3}{x-a} = 8$, and find the other root. (*E.M.E.U.*)

(17) If $s = A + Bn + Cn^2 + Dn^3$ and
$s = 0$ when $n = 0$,
$s = 1$ when $n = 1$,
$s = 5$ when $n = 2$,
and $s = 14$ when $n = 3$,
find A, B, C, D and the value of s when $n = 4$. (*E.M.E.U.*)

(18) Find the values of x and $\left(\frac{1}{y}\right)$ given that

$$5x - 2\left(\frac{1}{y}\right) = 11,$$
$$x + 3\left(\frac{1}{y}\right) = 9.$$

Then find the value of $\frac{xy + 1}{y}$. (*N.C.*)

(19) (i) Solve the equation $\frac{1}{3}(x + 1) = \frac{1}{2}(x - 1)$.
(ii) If $3p + 2q = 1{\cdot}6$ and $2p + 4q = 4{\cdot}8$, find the values of $(p + q)$ and $(7p + 2q)$. (*N.C.*)

(20) Solve the following equations—
(*a*) $\frac{1}{2}(x + 1) + \frac{3}{4}(x + 1) = 4$.
(*b*) $3x + 2y + 17 = 0$,
$5x + 3y + 27 = 0$. (*N.C.*)

(21) Solve the following simultaneous equations algebraically and graphically—

$$2x - 4(2 - y) = 2,$$
$$3x + 5(y - 2) = 4.$$ (*N.C.*)

(22) Find the values of $\frac{1}{x}$ and $\frac{1}{y}$ given that $\frac{4}{x} - \frac{1}{y} = 13$ and $\frac{3}{x} - \frac{2}{y} = 6$.
Then find the values of x and y and of $\frac{y - x}{y + x}$. (*N.C.*)

(23) (*a*) Find the values of d from the following equation—

$$2d^2 + 1{\cdot}2d - 6{\cdot}3 = 0.$$

(*b*) Given $P = A + \frac{B}{r^2}$, find the values of A and B if $P = 200$ when $r = 2{\cdot}5$ and $P = 2000$ when $r = 2$. (*U.L.C.I.*)

(24) (*a*) Factorize the expression $2x^2 - 7x + 6$, and find two values of x which will satisfy the equation $2x^2 - 7x + 6 = 0$.
(*b*) Solve the following equations—

$$2x + 3y = 10,$$
$$\frac{x}{15} = \frac{y}{30}.$$ (*U.L.C.I.*)

(25) (*a*) Show that $x - 5$ is a factor of $2x^3 - 3x^2 - 39x + 20$.
Find the other factors.
Hence solve the equation $2x^3 - 3x^2 - 39x + 20 = 0$.
(*b*) It takes 200 square tiles to pave a certain floor. If the tiles were an inch longer each way it would take only 128. Find the size of the tiles.
(*E.M.E.U.*)

(26) One of the three values for x for which the function $2x^3 + x^2 - 13x + 6$ has a zero value is -3. Find the other two. (*N.C.*)

(27) The values of the maximum and minimum stresses in the metal of a rivet due to a shear stress q and a tensile stress f, are given by the values of f in the equation $f(f - f_1) = q^2$. If $q = 4\frac{1}{2}$ tons per square inch and $f_1 = 3\frac{1}{2}$ tons per square inch, find the two values of f. (*U.E.I.*)

(28) Solve the following equations—

(*a*) $3x^2 - 10x + 5 = 0$,

(*b*) $x^2 - y^2 = 12$,

$x + y = 2$. (*U.L.C.I.*)

(29) Solve the equation $y^2 = 16x$, $3y = 4x - 16$. Verify the solution graphically. (*U.E.I.*)

(30) A consignment of 24 yd. of 4 in. and 16 yd. of 6 in. cast iron pipes costs £13 12s. A second consignment of 30 yd. of 4 in. pipes and 24 yd. of 6 in. pipes costs £18 15s. What is the cost per yard of each size of pipe? (*N.C.*)

(31) Solve the quadratic equations—

(*a*) $7x^2 - 3x = 160$.

(*b*) $(5x - 3)^2 - 7 = 44x - 5$. (*N.C.*)

(32) Solve the equations—

(*a*) $6x^2 - 19x + 10 = 0$.

(*b*) $3x^3 - 10x^2 + 9x - 2 = 0$, given that $x = 1$ is one root. (*N.C.*)

(33) Solve the equations—

(*a*) $3(2x - 1) - \dfrac{2 - 3x}{2} = 0{\cdot}5\,(x - 1)$.

(*b*) $1{\cdot}4x - \dfrac{3x - y}{5} = 2$,

$4x - (2x - 3y) = 7\frac{1}{2}$. (*N.C.*)

(34) (*a*) Simplify the following expression for H—

$$H = \frac{M}{2L(D - L)^2} - \frac{M}{2L(D + L)^2}.$$

Find also, from your simplified expression, the value for H when L is small compared with D so that L^2 may be neglected.

(*b*) When a body moves in a circular path of radius r with a linear speed V, the centrifugal force F is directly proportional to the square of the speed and inversely proportional to the radius. If $F = 24$ when $V = 12$ and $r = 2$, determine V when $F = 40$ and $r = 3$. (*U.L.C.I.*)

CHAPTER II

TRIGONOMETRY

Trigonometrical Ratios of Angles. In Chapter V, Book I, we dealt with the sine, cosine and tangent of angles of 90° and less, and with the relationships between these ratios. Three further ratios must now be introduced.

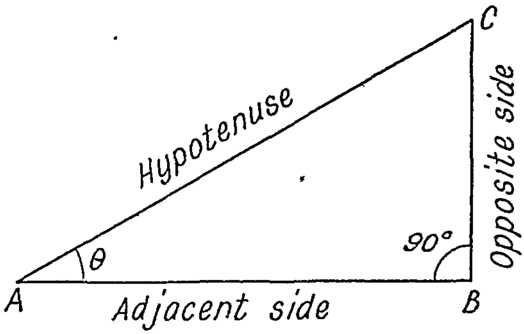


FIG. 2

Referring to Fig. 2, which shows a right-angled triangle, we have from Chapter V, Book I,

$$\sin\theta = \frac{CB}{CA}, \quad \cos\theta = \frac{AB}{CA}, \quad \tan\theta = \frac{CB}{BA}.$$

The three other ratios of the angle θ are called the *secant*, *cosecant* and *cotangent* of θ, or, more simply, sec θ, cosec θ and cot θ.

Now
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{CA}{AB}$$

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{CA}{CB}$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{BA}{CB}.$$

If these are compared with the preceding three ratios it will

be seen that the expression for sec θ is obtained by inverting the expression for cos θ, or, expressing this mathematically,

$$\sec\theta = \frac{1}{\cos\theta}.$$

Similarly
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

and
$$\cot\theta = \frac{1}{\tan\theta}.$$

Identities Involving these Ratios. A trigonometrical *identity* is, briefly, an equation which holds good for all angles.

Two identities which should be noted are

$$\sec^2\theta = 1 + \tan^2\theta$$

and
$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

These equations are true for *all* values of the angle θ and can be proved quite easily by reference to Fig. 2.

From this figure we have

$$\sec\theta = \frac{CA}{AB}, \text{ so that } \sec^2\theta = \frac{(CA)^2}{(AB)^2}.$$

Again

$$\tan\theta = \frac{CB}{AB}, \text{ so that } \tan^2\theta = \frac{(CB)^2}{(AB)^2}.$$

Then

$$1 + \tan^2\theta = 1 + \frac{(CB)^2}{(AB)^2}$$
$$= \frac{(AB)^2 + (CB)^2}{(AB)^2}.$$

Now, from Pythagoras's theorem relating to a right-angled triangle,

$$(AB)^2 + (CB)^2 = (CA)^2$$

so that we may write

$$1 + \tan^2\theta = \frac{(CA)^2}{(AB)^2} = \sec^2\theta.$$

In the same way

$$\operatorname{cosec}^2\theta = \frac{(CA)^2}{(CB)^2}$$

and $$1 + \cot^2\theta = 1 + \frac{(BA)^2}{(CB)^2} = \frac{(CB)^2 + (BA)^2}{(CB)^2}$$
$$= \frac{(CA)^2}{(CB)^2} = \operatorname{cosec}^2\theta.$$

Trigonometrical Identities. Utilizing the two fundamental identities given in the preceding paragraph, together with those derived in Book I (page 100), it is possible to prove a very large number of identities. Several examples of such proofs are given below. In these examples θ is *any* angle.

EXAMPLE 1. Prove that

$$\dagger \operatorname{cosec}\theta \tan\theta \cos\theta = 1.$$

In such cases it is often best to convert all the various ratios in terms of sines and cosines. Thus

$$\operatorname{cosec}\theta \tan\theta \cos\theta = \frac{1}{\sin\theta} \times \frac{\sin\theta}{\cos\theta} \times \cos\theta.$$

Cancelling the $\sin\theta$ with the $\sin\theta$ and the $\cos\theta$ with the $\cos\theta$ we obtain 1 on the right-hand side.

$$\therefore \qquad \operatorname{cosec}\theta \tan\theta \cos\theta = 1.$$

EXAMPLE 2. Prove that

$$\sin\theta \tan\theta + \cos\theta = \sec\theta.$$

Now $$\sin\theta \tan\theta + \cos\theta = \sin\theta \times \frac{\sin\theta}{\cos\theta} + \cos\theta$$
$$= \frac{\sin^2\theta}{\cos\theta} + \cos\theta$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}.$$

But, from Book I, page 101, $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \qquad \sin\theta \tan\theta + \cos\theta = \frac{1}{\cos\theta}$$
$$= \sec\theta.$$

† *Note.* When we write $\operatorname{cosec}\theta \tan\theta \cos\theta$ we mean $\operatorname{cosec}\theta \times \tan\theta \times \cos\theta$. The multiplication signs are often omitted, as in algebraical expressions.

EXAMPLE 3. Prove that

$$\sin\theta + \sin\theta\cot^2\theta + \frac{\tan\theta}{\cos\theta} = \frac{\sec^2\theta}{\sin\theta}.$$

Now

$$\sin\theta + \sin\theta\cot^2\theta + \frac{\tan\theta}{\cos\theta} = \sin\theta\,(1 + \cot^2\theta) + \frac{\tan\theta}{\cos\theta}$$

(to use the simplification $1 + \cot^2\theta = \operatorname{cosec}^2\theta$)

$$= \sin\theta\operatorname{cosec}^2\theta + \frac{\sin\theta}{\cos^2\theta} \qquad \left(\text{since } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

$$= \sin\theta\,\frac{1}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta} \qquad \left(\text{since } \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right)$$

$$= \frac{1}{\sin\theta} + \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos^2\theta} = \frac{1}{\sin\theta\cos^2\theta}$$

(since $\cos^2\theta + \sin^2\theta = 1$)

$$= \frac{\sec^2\theta}{\sin\theta}.$$

EXAMPLES IX

(1) Given that the sine of an angle A is $\frac{3}{7}$ calculate the value of cosec A. Sketch a right-angled triangle containing the angle A. Calculate the length of the unknown side and hence find the values of sec A and cot A.

(2) If $\cos\theta = \frac{12}{13}$ find the value of sec θ and also, using the method mentioned in Question 1, find the values of cosec θ and cot θ.

In Questions 3 to 13, prove the identities:

(3) $\dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$.

(4) $\dfrac{\sec^2 a}{\sec^2 a - 1} = \operatorname{cosec}^2 a$.

(5) $\dfrac{\sec\theta + \tan\theta}{(1 + \sin\theta)\cos\theta} = 1 + \tan^2\theta$.

(6) $\dfrac{\sec^2 A - \tan^2 A}{\sin A} = \sin A + \cos A\cot A$.

(7) $\dfrac{\tan A - \cot A}{\operatorname{cosec} A} \times \dfrac{1}{\tan A - 1} = \sin A + \cos A$.

(8) $\dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$.

(9) $\dfrac{1 - \cos A}{1 + \cos A} + \dfrac{1 + \cos A}{1 - \cos A} = 2 + 4\cot^2 A$.

(10) $\left(\frac{1-\cos A}{1+\cos A}\right)^{\frac{1}{2}} + \left(\frac{1+\cos A}{1-\cos A}\right)^{\frac{1}{2}} = 2 \operatorname{cosec} A.$

(11) $(1 + \tan A) \sin A = \frac{1}{\cos A - \sin A} - \frac{1}{\cos A}.$

(12) $\frac{\tan A}{\sec A - 1} + \frac{\tan A}{\sec A + 1} = 2 \operatorname{cosec} A.$

(13) $(\operatorname{cosec} A + \cos A)^2 + (\sec A + \sin A)^2 = (\operatorname{cosec} A \sec A + 1)^2.$

Expression of an Angle by reference to one of its Trigonometrical Ratios. There are occasions when we know one of the trigonometrical ratios of an angle, but do not know the value of the angle itself—possibly because trigonometrical tables are not available.

In such cases we can refer to the angle as, for example, "the angle whose sine is 0·5," or as "the angle whose cosine is 0·866." Another way of expressing these facts is to state that

$$\sin \theta = 0{\cdot}5$$

or

$$\cos \theta = 0{\cdot}866$$

where θ is the unknown angle.

Short ways of writing these expressions have been invented. The first is to use an index figure $^{-1}$ above the word "sin" or "cos." Thus

$\sin^{-1} 0{\cdot}5$ means "the angle whose sine is 0·5"

and

$\cos^{-1} 0{\cdot}866$ means "the angle whose cosine is 0·866."

$\sin^{-1} 0{\cdot}5$ does *not* mean $\frac{1}{\sin 0{\cdot}5}$ (see note below).

The second method of expressing an angle in terms of one of its trigonometrical ratios is to write the word "arc" instead of "the angle whose . . . is." In this method we have, for example,

$$\begin{aligned} \text{arc sin } 0{\cdot}5 &= 30° \\ \text{arc cos } 0{\cdot}866 &= 30° \\ \text{arc tan } 0{\cdot}5774 &= 30°. \end{aligned}$$

Note. As we have seen, we use $\sin^2 \theta$ to denote the square of $\sin \theta$ and, similarly, we use $\cos^3 \theta$ to denote the cube of $\cos \theta$.

For negative powers we use one of two forms—

(*a*) With bracket, such as $(\sin \theta)^{-2}$ and $(\cos \theta)^{-3}$ or, (*b*) as fractions, such as $\frac{1}{\sin^2 \theta}$ and $\frac{1}{\cos^3 \theta}$. We avoid using a notation of the type $\sin^{-2} \theta$ or $\cos^{-3} \theta$ on account of the special meaning of $\sin^{-1} \theta$, etc.

Trigonometrical Equations. Such equations involve the trigonometrical ratios of a particular angle whose value is to be found from the equation given. They differ from the identities which we have spoken about in a preceding paragraph in that they hold good only for *a particular angle or angles*, and are *not* generally true for all angles, as are identities.

EXAMPLE 1. Find the value of an angle θ given that

$$3 \sin \theta = 4 \cos \theta.$$

In all trigonometrical equations it is necessary first of all to convert the equation into one which involves only one ratio of the angle to be found. This can be done by utilizing the known relationships between the various ratios.

Thus, in this example, if we divide each side by $\cos \theta$ we have

$$\frac{3 \sin \theta}{\cos \theta} = 4.$$

Now we know that, for any angle θ, $\frac{\sin \theta}{\cos \theta} = \tan \theta$, so that we may write

$$3 \tan \theta = 4.$$

From which $\tan \theta = \frac{4}{3} = 1{\cdot}333.$

Referring to the tables of tangents we find that $\theta = 53° \ 8'$. (Note that instead of finding θ from the tables we could have stated the answer as

$$\theta = \tan^{-1} \frac{4}{3};$$

although it is better to give the value of θ in degrees as above.)

EXAMPLE 2. If $2 \sec^2 a - 1 = 2 \tan a \ (\tan a + 1)$ find the value of a.

Substituting $1 + \tan^2 a$ for $\sec^2 a$ we have

$$\begin{aligned} 2(1 + \tan^2 a) - 1 &= 2 \tan a (\tan a + 1) \\ \text{or} \quad 2 + 2 \tan^2 a - 1 &= 2 \tan^2 a + 2 \tan a \\ 1 &= 2 \tan a \\ \tfrac{1}{2} &= \tan a; \\ \therefore \ a &= \tan^{-1} \tfrac{1}{2} \\ &= 26° \ 34' \end{aligned}$$

EXAMPLES X

(1) If $\theta = 4°$ find the values of—

$$\sin^2\theta,\ \cos^3\theta,\ \tan 4\theta,\ \sin \tfrac{1}{2}\theta.$$

(2) Express in degrees and minutes the following angles—

$$\sin^{-1} 0{\cdot}8,\ \cos^{-1} \tfrac{5}{13},\ \text{arc tan } 0{\cdot}75,\ \text{arc cos } 0{\cdot}5.$$

Solve the following—

(3) $3 \sec\theta = 5 \tan\theta.$
(4) $2/(1 + 2 \sin\theta) = 1.$
(5) $\sin\theta \tan\theta + \sec\theta + \cos\theta = 4.$
(6) $\sec^2\theta + 3 \tan^2\theta = 5.$
(7) $5 \cos^2\theta - 8 \cos\theta + 3 = 0.$
(8) $2 \sin\theta = \sqrt{6 \cos\theta}.$
(9) $6 \sin^2\theta - 5 \sin\theta + 1 = 0.$
(10) $6 \cos^2\theta - 11 \cos\theta + 4 = 0.$
(11) $9 \sin^2\theta - 6 \sin\theta + \cos^2\theta = 0.$
(12) $8 \sin^2\theta - 7 \sin\theta + 2 \cos^2\theta = 0.$
(13) $2 \sin^2\theta - 3 \sin\theta \cos\theta + \cos^2\theta = 0.$
(14) $12 \sin^2\theta - 13 \sin\theta \cos\theta + 3 \cos^2\theta = 0.$
(15) $5 \cos^2\theta - 5 \cos\theta \sin\theta + 1 = 0.$
(16) $\sec^2\theta = 4 \tan\theta - 2.$
(17) $\cot^2\theta = 3(\text{cosec}\,\theta - 1).$
(18) $2 - \cot\theta = \text{cosec}\,\theta.$

Angles Greater than 90°. Up to the present we have dealt only with the trigonometrical ratios of *acute* angles, i.e. of

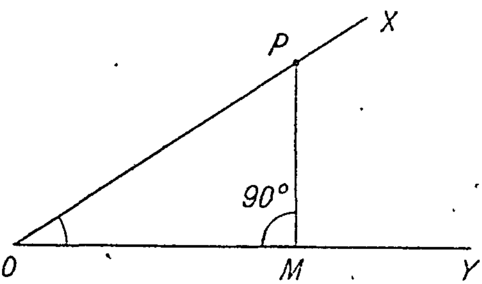


FIG. 3

angles less than 90°. We now have to consider angles greater than 90°.

Now, when we considered, in Chapter V, Book I, the trigonometrical ratios of an acute angle, such as that shown in Fig. 3, we first of all drew a line from any point P on OX at right angles to the line OY.

If, however, the angle is *obtuse* (i.e. greater than 90°), it is not always possible to draw a straight line from any point P to meet OY at right angles. In such cases (e.g. 130° or 240°) we

draw a straight line, PM, from P at right angles to OY produced backwards through O to meet it at M. We define the trigonometrical ratios by the *same* equalities as we used for the acute angle, namely

$$\sin\theta = \frac{MP}{OP}$$

$$\cos\theta = \frac{OM}{OP}$$

$$\tan\theta = \frac{MP}{OM}$$

except that we give the lengths MP and OM the signs employed in graphical work. Thus MP is positive if P is above M and

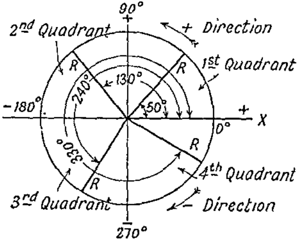


FIG. 4

negative if it is below; OM is positive if M is to the right of O and negative if it is to the left (in fact MP and OM are the y and x of graphical work). OP is *always* positive (see Figs. 6, 7, and 8).

Using now the conventional graphical figure we take two axes at right angles and a radius R rotating in the anti-clockwise direction about O from the position OX. The angle which R makes with OX is the measure of the rotation, and its extremity

traces out the circle shown in Fig. 4. These axes divide the figure into four quadrants, and if

R lies in the first quadrant, the angle is between						0° and 90°
,,	,,	second	,,	,, ,,	,,	90° and 180°
,,	,,	third	,,	,, ,,	,,	180° and 270°
,,	,,	fourth	,,	,, ,,	,,	270° and 360°.

In Fig. 4 the radius R is drawn in four representative positions, the angles represented being 50°, 130°, 240°, and 330°.

For an angle greater than 360° we make R rotate through one or more complete revolutions as required. Thus for 500° R rotates through one complete revolution and a further 140°, i.e. R lies finally in the second quadrant; for 920° R rotates through two complete revolutions and a further 200°, i.e. R lies finally in the third quadrant.

We obtain *negative* angles by making R rotate backwards, thus

> for $-400°$ R rotates *backwards* through one complete revolution and a further 40°, i.e. R lies finally in the fourth quadrant.

The angles 500°, 920°, and $-400°$ are represented in Fig. 5 (A), (B), and (C).

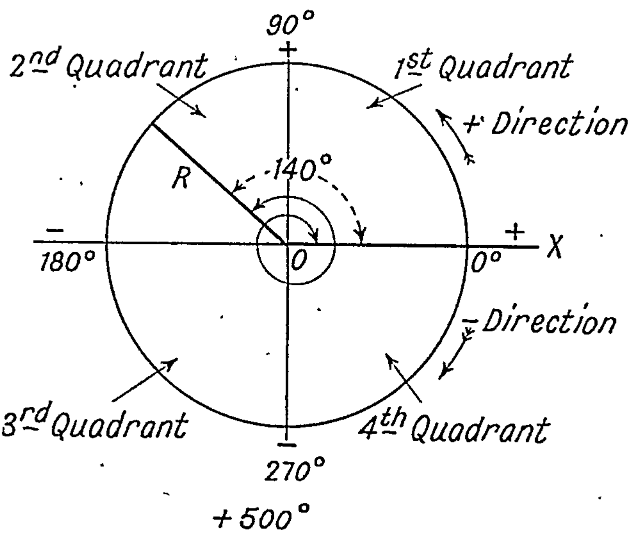


FIG. 5 (A)

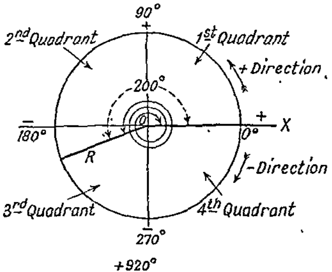


FIG. 5 (B)

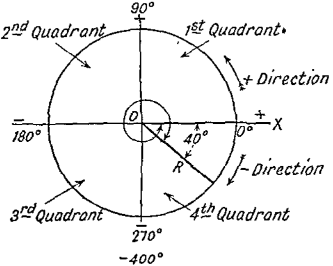


FIG. 5 (C)

Angles Between 90° and 180°. Referring to Fig. 6 the angle between the rotating radius R and line OX lies between 90° and 180°.

We draw a line from the outer end P of the line R at right angles *to the horizontal axis.* (It should be noted here that, whatever the value of the angle considered, we *always* draw lines from the outer end of R at right angles to the *horizontal* axis.) Let the angle XOP considered be θ.

Then $$\sin\theta = \frac{MP}{OP}.$$

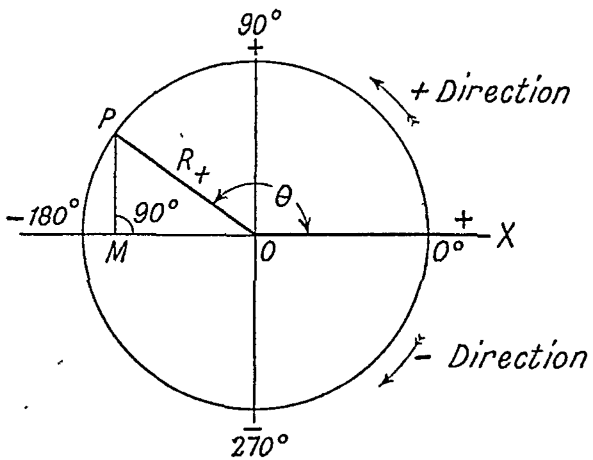


FIG. 6

We now insert the signs (positive or negative) of these lengths, the radius R *always being reckoned positive whatever its position.*

Then $$\sin\theta = \frac{+MP}{+OP} = +\frac{MP}{OP}.$$

Now in the triangle POM the ratio $\frac{MP}{OP}$ is obviously the sine of the acute angle POM. Again this acute angle is $(180 - \theta)$ since MOX is a straight line. Therefore we can write

$$\sin\theta = +\sin(180 - \theta).$$

For example,

$$\begin{aligned} \sin 120 &= +\sin(180 - 120) \\ &= +\sin 60 \\ &= +0{\cdot}8660. \end{aligned}$$

Again $\cos\theta = \dfrac{OM}{OP}$ or, inserting signs,

$$\cos\theta = \frac{-OM}{+OP} \quad \text{(since } OM \text{ is a negative length)}$$
$$= -\frac{OM}{OP}$$
$$= -\cos(180 - \theta),$$

since referring to triangle POM the ratio $\dfrac{OM}{OP} = \cos P\hat{O}M = \cos(180 - \theta)$

$$\tan\theta = \frac{MP}{OM} \text{ or, inserting signs,}$$
$$\tan\theta = \frac{+MP}{-OM} = -\frac{MP}{OM}$$
$$= -\tan(180 - \theta).$$

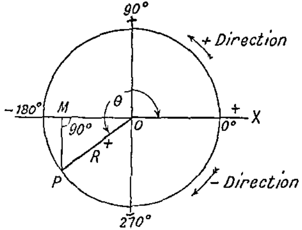


FIG. 7

Summarizing: for an angle between 90° and 180° we have

$$\sin\theta = +\sin(180 - \theta)$$
$$\cos\theta = -\cos(180 - \theta)$$
$$\tan\theta = -\tan(180 - \theta).$$

Angles Between 180° and 270°. Referring to Fig. 7 we have, inserting signs as before,

$$\sin\theta = \frac{-MP}{+OP} = -\frac{MP}{OP}.$$

Now in triangle POM the ratio $\frac{MP}{OP}$ is the sine of the acute angle POM, i.e. the sine of $\theta - 180$.

$$\therefore \quad \sin\theta = -\sin(\theta - 180)$$

$$\cos\theta = \frac{-OM}{+OP} = -\frac{OM}{OP}$$

$$= -\cos(\theta - 180)$$

$$\tan\theta = \frac{-MP}{-OM} = +\frac{MP}{OM}$$

$$= +\tan(\theta - 180).$$

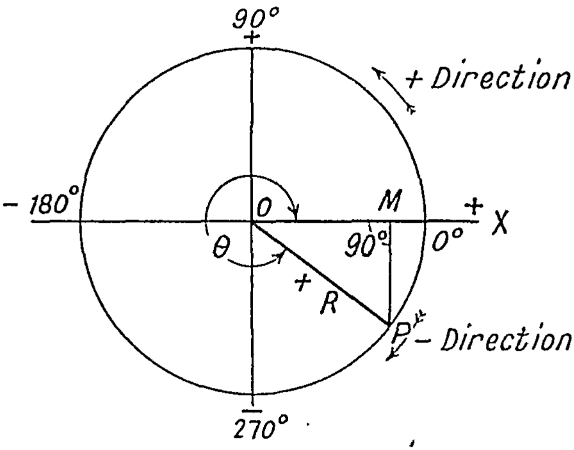


FIG. 8

For example, $\cos 240 = -\cos(240 - 180)$

$$= -\cos 60$$

$$= -0{\cdot}5000.$$

Summarizing: for angles between 180° *and* 270° *we have*

$$\sin\theta = -\sin(\theta - 180)$$

$$\cos\theta = -\cos(\theta - 180)$$

$$\tan\theta = +\tan(\theta - 180).$$

Angles Between 270° and 360°. Referring to Fig. 8 and proceeding as before we have

$$\sin\theta = \frac{-MP}{+OP} = -\frac{MP}{OP}.$$

In triangle POM the ratio $\frac{MP}{OP}$ is the sine of acute angle POM, i.e. the sine of $(360 - \theta)$.

Hence
$$\sin \theta = -\sin (360 - \theta).$$

Similarly
$$\cos \theta = \frac{+OM}{+OP} = +\frac{OM}{OP}$$
$$= +\cos (360 - \theta)$$

and
$$\tan \theta = \frac{-MP}{+OM} = -\frac{MP}{OM}$$
$$= -\tan (360 - \theta).$$

For example,
$$\tan 330 = -\tan (360 - 330)$$
$$= -\tan 30$$
$$= -0{\cdot}5774.$$

Summarizing: for angles between $270°$ *and* $360°$ *we have*

$$\sin \theta = -\sin (360 - \theta)$$
$$\cos \theta = +\cos (360 - \theta)$$
$$\tan \theta = -\tan (360 - \theta).$$

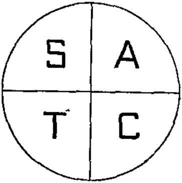


FIG. 9

A convenient method of summarizing the above, from the point of view of the *signs* of the various ratios in the different quadrants, is by the diagram shown in Fig. 9. The letters in the quadrants indicate which of the ratios is positive when the angle is such that R lies in that quadrant. Thus in

First Quadrant.	*All ratios are positive.*
Second ,,	*Sine only is positive.*
Third ,,	*Tangent only is positive.*
Fourth ,,	*Cosine only is positive.*

In the previous work, the changes in angles have been by even numbers of right angles (180°, 360°). In Questions 10 and 12, the student will be asked to make changes by odd numbers of right angles (90°, 270°); and, if he summarizes his results, he will find the following convenient rule: From a change by an *even* number of right angles, the *name* of the ratio is unaltered; but from a change by an *odd* number of right angles, the *name* of the ratio is altered (sine becomes cosine, cosine becomes sine, tan becomes cot, etc.).

EXAMPLES XI

(1) Find the sines of—

$$145°, 230°, 290°, 575°.$$

(2) Find the cosines of—

$$160°, 250°, 320°, 1092°, -127°.$$

(3) Find the tangents of—

$$135°, 260°, 335°, 880°, -250°.$$

(4) Find the secants and cosecants of—

$$195°, 340°, 790°, -426°.$$

(5) State all the angles between 0° and 720° whose sine is $\frac{1}{2}$.

(6) Find the values of $\sin \frac{5}{2}\pi$, $\cos \frac{7}{3}\pi$, $\tan \frac{14}{5}\pi$.

(7) Find the values of $\sec(-\frac{7}{4}\pi)$, $\operatorname{cosec}(-\frac{5}{3}\pi)$, $\cot(-27\pi)$.

(8) If $\sin\theta = -\frac{4}{5}$ and θ lies between 180° and 270° find the values of $\sec\theta$ and $\tan\theta$.

In Questions 9 to 14, the angle A is less than 90°.

(9) Show, by drawing, in what quadrants the revolving radius lies when the angle represented is: $(180 + A)$, $(180 - A)$, $(270 - A)$.

(10) Express the sine, cosine and tangent of each of the angles $(90 + A)$ and $(270 - A)$ in terms of trigonometrical ratios of the angle A.

(11) Express the sine, cosine and tangent of each of the angles $(180 - A)$ and $(360 - A)$ in terms of trigonometrical ratios of the angle A.

(12) Express the cosecant, secant and cot of each of the angles $(90 + A)$ and $(270 + A)$ in terms of the trigonometrical ratios of the angle A.

(13) Prove the identity: $\dfrac{1 - \cos A}{\sin(270 - A)} + \sec(360 - A) = 1.$

(14) Prove the identity:

$$\tan(270 + A) - \cot(180 + A) = -\sec(90 - A)\operatorname{cosec}(90 + A).$$

(15) Without the use of tables, prove that

$$\cos 19° + 2\sin 109° + 6\cos 161°\cos 300° = 0.$$

(16) Without the use of tables, prove that

$$\tan 130° - \cot 40° - 2\tan 230°\tan 325° = 0.$$

The Compass—Bearings. In surveying work it is often very valuable to know the direction in which a certain point lies relative to another point. Calling the latter point A and the former point B, we speak of this relative direction as the *bearing* of B from A. Thus, if point B lies directly to the north of

point A we say that "B bears due north from A." After carrying out a survey in which various distances have been measured, and the relative bearings of various points observed, we then utilize trigonometry to make calculations of other distances which have not been measured.

The compass is used by surveyors in expressing the relative directions, or bearings, of a number of points. Fig. 10 shows

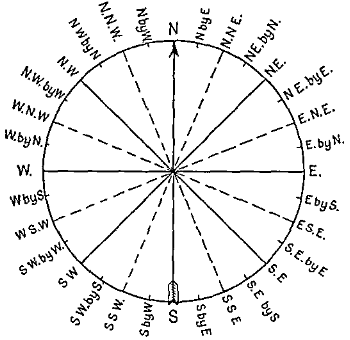


Fig. 10

the *points* of the compass. The circle is divided into 32 equal parts, so that there are 32 points, the angle between two adjacent points being $(\frac{360}{32})^\circ$, i.e. 11° 15′. North, South, East and West are called the *Cardinal Points*.

A direction which does not lie exactly along any one of the points of the compass is expressed in terms of the cardinal points as follows—

N. 40° W., or S. 20° E., and so on.

These two directions are shown in Fig. 11, and are, in words, "40° west of north" and "20° east of south." Referring to the

points B and C in this figure we should say that "B bears N. 40° W. from point A" and that "C bears S. 20° E. from point A."

Another method of stating bearings is the one now used throughout His Majesty's Forces. A direction is given by the angle it makes with the North measured through the East. Thus in the figure the bearings of B and C from A are 320° and 160° respectively, and the bearing of A from C is 340°.

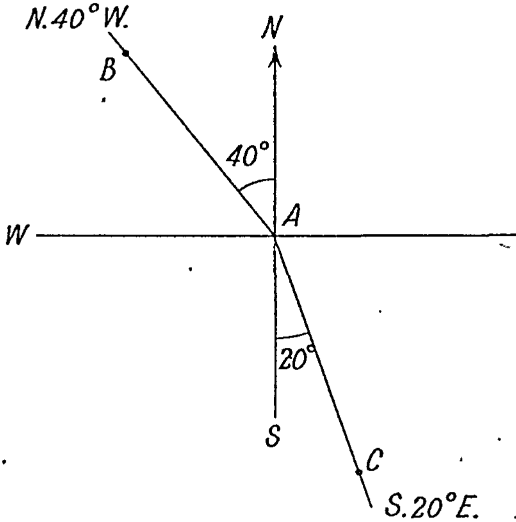


FIG. 11

EXAMPLE. The bearing of a ship is N. 30° E. from a point A on the coast. From another point B on the coast, 3 miles due east of A, the ship bears N. 60° W.

Calculate the distances of the ship from A and B.

Fig. 12 illustrates this problem.

From the information given it is clear

that $\quad$ angle $CAB = 90° - 30° = 60°$

and $\quad$ angle $CBA = 90° - 60° = 30°$.

Thus $\quad$ angle $ACB = 90°$.

Hence
$$\frac{CB}{3} = \sin 60° = 0{\cdot}866,$$

$$\therefore \quad CB = 3 \times 0{\cdot}866$$
$$= 2{\cdot}598 \text{ miles.}$$

Again $\frac{CA}{3} = \cos 60° = 0{\cdot}5,$

$\therefore \quad CA = 3 \times 0{\cdot}5$
$= 1{\cdot}5$ miles.

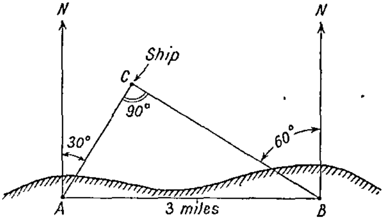


FIG. 12

EXAMPLES XII

(1) Draw the directions—
N. 37° E., S. 48° W.; S. 57° E.; N. 64° W.

(2) If a point B bears N 25° E. from A and N. 44° W. from C, and if C bears S. 72° E. from A, make a drawing to show the relative bearings of the three points.

(3) A point B lies 2 miles to the S.W. of a point A and a point C lies 3 miles to the N.W. of point B. Calculate the bearing of C from A and the distance between C and A.

(4) The bearing of an object from an observer is N. 22½° W. The observer then walks 800 yd. in a direction N. 67½° E. and the bearing of the same object is N. 53° W. Calculate the distances of the object from the observer before and after his walk.

(5) From a point A, the bearing of B, 3 miles distant from A, is 20°, and the bearing of C is 65°; from B the bearing of C is 105°. Make a scale drawing to find the distance of C from B.

(6) From a point A, the bearing of B, 10 miles distant from A, is 47°; through A a straight road runs on the bearing 98°. Calculate the distance of B from the road.

(7) From a point A, the bearing of B, 50 miles distant from A is 350°: on what bearing should an aeroplane fly, starting from A, if the nearest point of its path from B is to be 15 miles? (*Note.* There are two solutions.)

(8) A ship sails 10 miles on the bearing 60° and then 10 miles on the bearing 150° What is then its bearing from the starting point?

(9) If in Question 8 the bearings are 10° and 310°, what is then the bearing of the starting point from the ship?

(10) A ship sails for 20 miles on the bearing 205° and then for 12 miles on the bearing 115°. What is then its bearing from the starting point?

(11) From a point A the bearings of B and C, both 10 miles from A, are 255° and 287°. Calculate the distance and bearing of C from B.

The "Sine" Formula. This formula holds good for *all triangles, whether right-angled or not.*

It states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

where a, b and c are the sides opposite to the angles A, B and C respectively.

The formula is very commonly used for the solution of triangles which are not right-angled, but it should be noted at the outset that at the present stage *it can only be used when an angle and the side opposite to it are known.*

For example if we are given sides a and b and angle A in a triangle, we can immediately proceed to find the angle B since

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

or

$$\sin B = \frac{b}{a}\sin A,$$

all three of the quantities on the right-hand side being known.

Continuing the solution of the triangle, we then find angle C by subtracting $(A + B)$ from 180°, thus

$$C = 180 - (A + B).$$

Finally side c is found from the equation

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

or

$$c = \frac{a}{\sin A} \times \sin C.$$

Proof of the Sine Formula. Referring to Fig. 13, ABC is any triangle (not necessarily right-angled). First draw a line CD, of length p, from C at right angles to AB.

Then in triangle ACD.

$$\frac{p}{b} = \sin A$$

or

$$p = b \sin A.$$

Again, in triangle BCD,

$$\frac{p}{a} = \sin B$$

or $$p = a \sin B.$$

Thus $$b \sin A = a \sin B$$

(since both are equal to p).

Hence, dividing each side by ($\sin A \times \sin B$), we have

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

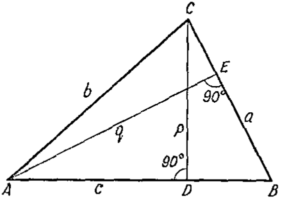


FIG. 13

Next draw AE (of length q) at right angles to CB.

Then, in triangle CAE,

$$\frac{q}{b} = \sin C$$

or $$q = b \sin C.$$

In triangle BAE

$$\frac{q}{c} = \sin B$$

or $$q = c \sin B.$$

$\therefore$ $$b \sin C = c \sin B$$

or $$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

But, from above,

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

EXAMPLE. From a ship which is sailing north-east at 16 m.p.h. a lighthouse bears due north. Half an hour later the lighthouse bears S. 70° W.; how far was the ship from the lighthouse when the first bearing was taken?

Fig. 14 illustrates the example. Since the lighthouse B

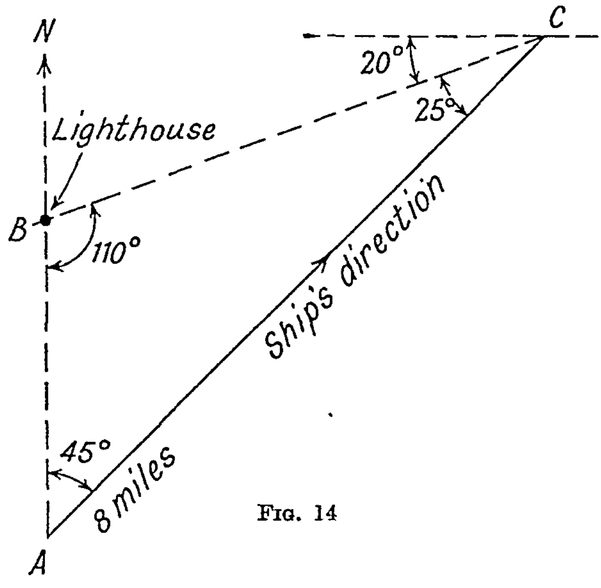


FIG. 14

bears S. 70° W. from the second position C of the ship the angle $BCA = 25°$.

$\therefore$ Angle $ABC = 180° - 45° - 25° = 110°$.

The distance travelled by the ship in half an hour $= 8$ miles $= AC$.

We want to find the length AB. Using the sine formula we have

$$\frac{8}{\sin 110°} = \frac{AB}{\sin 25°}$$

$$\therefore \qquad AB = \frac{8 \times \sin 25^\circ}{\sin 110^\circ}.$$

Now $\sin 110^\circ = \sin (180^\circ - 110^\circ) = \sin 70^\circ$;

$$\therefore\ AB = \frac{8 \times 0{\cdot}4226}{0{\cdot}9397} = 3{\cdot}6 \text{ miles (to two significant figures).}$$

The method of the following example is sometimes also useful as an alternative to that of the sine formula. In it we work from one part of the figure to another part by means of right-angled triangles.

Suppose, using the Fig. 14 of the previous example, we had been given that $AB = 3{\cdot}6$ miles, angle $BAC = 45^\circ$ and angle $ABC = 110^\circ$, and had been asked to find the perpendicular distance of C from AB.

Suppose the perpendicular distance from C to AB (produced) meets it at D, and that $CD = x$ miles.

From right-angled triangle ADC,

$$AD = x \cot 45^\circ.$$

From right-angled triangle BDC,

$$BD = x \cot (180 - 110)^\circ = x \cot 70^\circ.$$

$\therefore$ Subtracting,

$$AD - BD = x\{\cot 45^\circ - \cot 70^\circ\}$$

or
$$3{\cdot}6 = x\{1 - 0{\cdot}3640\} = 0{\cdot}6360x$$

and
$$x = 3{\cdot}6/0{\cdot}6360 = 5{\cdot}66 \text{ miles.}$$

EXAMPLES XIII

Solve the triangles in which—

(1) $\hat{A} = 50^\circ$, $\hat{B} = 35^\circ$; $b = 2{\cdot}5$ in.

(2) $\hat{C} = 43^\circ$; $a = 12{\cdot}5$ ft.; $c = 10$ ft. and check graphically. (*Note.* Two solutions; if $\sin A = 0{\cdot}8525$ either $A = 58^\circ\ 29'$ or $A = 121^\circ\ 31'$.)

(3) $\hat{A} = 58^\circ$; $\hat{B} = 73^\circ$; $c = 2{\cdot}1$ miles.

(4) In finding the position of a certain point C a surveyor takes readings from the two ends of a line AB which is 1200 yd. long. He finds that the angle $CBA = 37^\circ\ 35'$ and angle $CAB = 48^\circ\ 17'$.
Calculate the distances CA and CB.

(5) From a point A the bearing of a tower is 320°, and from a point B, 400 yd. from A, the bearing is 255°. If the bearing of B from A is 30°, find the distance of the tower from A.

(6) From a point X, the bearings of points A and B are 345° and 12°: from a point Y their bearings are both 70°. If Y is 300 yd. from X on the bearing 270°, find the distance of B from A.

(7) An aeroplane is flying at a constant height above the ground and its speed is 120 m.p.h. An observer, situated on the ground immediately under the line of flight, finds the angle of elevation of the plane to be 43°. Two

minutes later the aeroplane has passed over the observer's head and the angle of elevation is 68°. What was the actual distance, in miles, between plane and observer at the time of the first observation?

(8) A vertical post AB stands with its foot A on horizontal ground. The line ACD on the ground is marked. From C the elevation of B is 53°; from D the elevation of B is 42° 6′, and CD is 10 ft. Find the height of B above A.

(9) A man walks along a level road towards a telegraph post. At one point he notices that the elevation of the top of the post is 36° 18′, while at the point 12 ft. nearer the post it is 50° 12′. Find how high the top of the post is above the man's eye.

(10) A man on a cliff observes a boat at an angle of depression of 28°. The boat is making for the shore immediately beneath him, and 3 min. later the angle of depression is 62°. After how much longer will it reach the shore?

(11) From one end of a yacht the angle of elevation of the top of the mast is 73° 24′, and from the other it is 59° 36′. If the length of the yacht is 60 ft. how high is the mast?

Compound Angles. By the term *compound angle* is meant an angle which is made up of the sum of two or more angles.

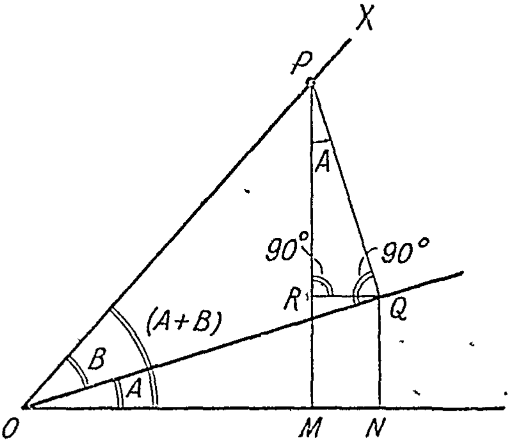


FIG. 15

(Some of these angles may be negative, in which case the compound angle may be looked upon as a difference of angles rather than as a sum.)

There are certain important formulae connected with such compound angles which enable one to express their trigonometrical ratios in terms of those of their component angles.†

Trigonometrical Ratios of an Angle $A + B$. Consider a compound angle $A + B$ made up of two component angles A and B added together as in Fig. 15.

P is *any* point on the line OX.

† In the proofs all the angles are supposed to be less than 90°; the results, however, are quite general.

Construction lines are drawn as shown to assist in deriving an expression for the trigonometrical ratios of the compound angle.

Now RQ is parallel to ON, and thus

$$R\hat{Q}O = Q\hat{O}N = A$$

$$\therefore \quad R\hat{Q}P = 90^\circ - A$$

$$\therefore \quad R\hat{P}Q = A.$$

(i) Then

$$\sin(A + B) = \frac{PM}{OP} \text{ (from triangle } POM\text{)}$$

$$= \frac{PR + RM}{OP}.$$

But $RM = QN$ (since they are opposite sides of the rectangle $MRQN$).

$$\therefore \quad \sin(A + B) = \frac{PR + QN}{OP}$$

$$= \frac{PR}{OP} + \frac{QN}{OP}.$$

Now $\quad PR = PQ\cos A \quad$ (from triangle RPQ)

and $\quad QN = OQ\sin A \quad$ (from triangle QON)

$$\therefore \quad \sin(A + B) = \frac{PQ\cos A}{OP} + \frac{OQ\sin A}{OP}.$$

Again $\quad \dfrac{PQ}{OP} = \sin B \quad$ (from triangle POQ)

and $\quad \dfrac{OQ}{OP} = \cos B \quad$ (from triangle POQ)

$$\therefore \quad \sin(A + B) = \sin B\cos A + \cos B\sin A$$

or, as it is more usually written,

$$\boldsymbol{\sin(A + B) = \sin A\cos B + \cos A\sin B.}$$

(ii) $$\cos(A + B) = \frac{OM}{OP} \text{ (from triangle } POM\text{)}$$

$$= \frac{ON - MN}{OP}.$$

But $\qquad MN = RQ;$

$$\therefore \qquad \cos(A+B) = \frac{ON - RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}.$$

Now $\qquad ON = OQ \cos A \quad$ (from triangle QON)

and $\qquad RQ = PQ \sin A \quad$ (from triangle RPQ)

$$\therefore \qquad \cos(A+B) = \frac{OQ\cos A}{OP} - \frac{PQ\sin A}{OP}.$$

Again $\qquad \dfrac{OQ}{OP} = \cos B \quad$ (from triangle $P\dot{O}Q$)

and $\qquad \dfrac{PQ}{OP} = \sin B \quad$ (from triangle POQ)

$$\therefore \qquad \boldsymbol{\cos(A+B) = \cos A \cos B - \sin A \sin B.}$$

(iii) $\qquad \tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$\left(\text{As shown in Book I, when } \theta \text{ is } \textit{any} \text{ angle } \tan\theta = \dfrac{\sin\theta}{\cos\theta}\right).$

$$\therefore \qquad \tan(A+B) = \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing both numerator and denominator of the right-hand side by $\cos A \cos B$ we have

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

or

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}.$$

After cancelling

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Remembering that

$$\frac{\sin A}{\cos A} = \tan A \text{ and } \frac{\sin B}{\cos B} = \tan B$$

we have

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Trigonometrical Ratios of an Angle $2A$. If we take a special case of the compound angle $(A + B)$ in which $A = B$, the compound angle becomes $A + A = 2A$.

Then, substituting A for B in the formulae derived in the preceding paragraph, we have

(i) $$\sin 2A = \sin (A + A)$$
$$= \sin A \cos A + \cos A \sin A$$

or $$\sin 2A = 2 \sin A \cos A.$$

(ii) $$\cos 2A = \cos (A + A)$$
$$= \cos A \cos A - \sin A \sin A$$

or $$\cos 2A = \cos^2 A - \sin^2 A.$$

(iii) $$\tan 2A = \tan (A + A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

or $$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

The formula for $\cos 2A$ can be extended from the fact that, for any angle A, $\cos^2 A + \sin^2 A = 1$.

We can substitute either $(1 - \sin^2 A)$ for $\cos^2 A$ or $(1 - \cos^2 A)$ for $\sin^2 A$.

Using these substitutions we have

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= (1 - \sin^2 A) - \sin^2 A$$

or $$\cos 2A = 1 - 2 \sin^2 A.$$

Again,

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= \cos^2 A - (1 - \cos^2 A)$$

or $$\cos 2A = 2 \cos^2 A - 1.$$

Trigonometrical ratios of angles such as $3A$, $4A$, etc., can

be derived in terms of the ratios of the single angle A by methods similar to the above, writing for $3A$ the expression $(A + 2A)$; for $4A$ the expression $(A + 3A)$, and so on.

Trigonometrical Ratios of an Angle $A - B$. In Fig. 16 a compound angle $(A - B)$ is shown, with construction lines drawn as before, P being *any* point on the line OX.

Since lines OM and QR are parallel

$$\hat{A} + O\hat{Q}R = 180°$$

$$\therefore \quad O\hat{Q}R = 180° - \hat{A}.$$

But $$O\hat{Q}P = 90°$$

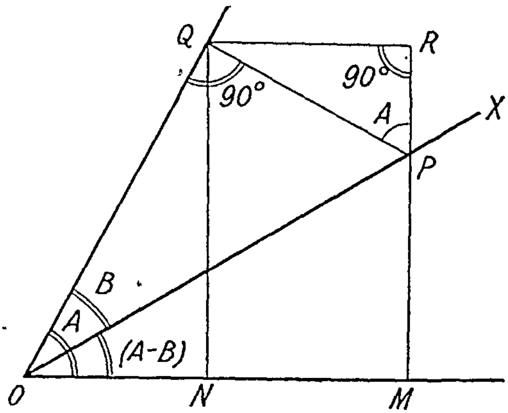


FIG. 16

$$\therefore \quad P\hat{Q}R = 90° - \hat{A}$$

$$\therefore \quad Q\hat{P}R = \hat{A}.$$

(i) Now $\sin (A - B) = \dfrac{PM}{OP}$ (from triangle POM)

$$= \frac{RM - RP}{OP} \text{ (since } PM = RM - RP\text{).}$$

Lines RM and QN are equal since they are opposite sides of the rectangle $NQRM$.

$$\therefore \quad \sin (A - B) = \frac{QN - RP}{OP}.$$

But $\quad QN = OQ \sin A$ (from triangle QON)

and $\quad RP = PQ \cos A$ (from triangle QPR)

$$\therefore \quad \sin(A - B) = \frac{OQ \sin A - PQ \cos A}{OP}$$

$$= \frac{OQ \sin A}{OP} - \frac{PQ \cos A}{OP}.$$

Again $\quad \dfrac{OQ}{OP} = \cos B$ (from triangle QOP)

and $\quad \dfrac{PQ}{OP} = \sin B$ (from triangle QOP)

$\therefore$ ***sin (A − B) = sin A cos B − cos A sin B.***

(ii) $\cos(A - B) = \dfrac{OM}{OP}$ (from triangle POM)

$$= \frac{ON + NM}{OP} \text{ (since } OM = ON + NM\text{)}$$

$$NM = QR$$

$$\therefore \quad \cos(A - B) = \frac{ON + QR}{OP}.$$

But $\quad ON = OQ \cos A$ (from triangle QON)

and $\quad QR = PQ \sin A$ (from triangle QPR)

$$\therefore \quad \cos(A - B) = \frac{OQ \cos A + PQ \sin A}{OP}$$

$$= \frac{OQ \cos A}{OP} + \frac{PQ \sin A}{OP}.$$

Again $\quad \dfrac{OQ}{OP} = \cos B$ (from triangle QOP)

and $\quad \dfrac{PQ}{OP} = \sin B$ (from triangle QOP)

$\therefore$ ***cos (A − B) = cos A cos B + sin A sin B.***

(iii) $$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}.$$

Dividing numerator and denominator by $\cos A \cos B$ we have

$$\tan (A - B) = \frac{\dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B + \sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} + \dfrac{\sin A \sin B}{\cos A \cos B}}$$

or, after cancelling,

$$\tan (A - B) = \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$\therefore \quad \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

EXAMPLES XIV

Tables must not be used in this set of examples: all necessary data are given in Question 1.

(1) Given that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$;

$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$;

$\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$.

Prove, by using the formulae for compound angles, that—

(i) $\sin 75° = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$;

(ii) $\cos 15° = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$;

(iii) $\tan 105° = \dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$;

(iv) $\sin 90° = 1$.

(2) Given than $\tan 19° 48' = 0{\cdot}3600$ and that $\tan 36° 30' = 0{\cdot}7400$, find—

(i) $\tan 39° 36'$;
(ii) $\tan 73°$;
(iii) $\tan 56° 18'$.

(3) If $\cos A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$, calculate the values of $\sin (A + B)$ and $\cos (A - B)$.

(4) If $\sin A = \frac{12}{13}$ and $\sin B = \frac{8}{17}$, calculate the values of $\sin (A - B)$ and $\cos (A + B)$.

(5) If $\sin A = \frac{7}{25}$, calculate the values of $\sin 2A$ and $\cos 2A$.

(6) If $\tan A = 1{\cdot}2$ and $\tan B = 0{\cdot}5$, calculate the values of $\tan (A + B$ and $\tan (A - B)$.

(7) Evaluate $\dfrac{\tan 83 - \tan 38}{1 + \tan 83 \tan 38}$.

(8) Evaluate $\dfrac{\tan 17 + \tan 13}{1 - \tan 17 \tan 13}$.

(9) Evaluate $\sin (10 + x) \cos (20 - x) + \cos (10 + x) \sin (20 - x)$.

(10) Evaluate $\cos (75 + x) \cos (30 + x) + \sin (75 + x) \sin (30 + x)$.

(11) Prove that $\dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$.

(12) Prove $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$.

(13) Prove that
$$\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B.$$
and check this when $A = 45°$ and $B = 30°$.

(14) Prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$.

(15) Prove that
$$\sin 3y + 4 \sin (y - 60) \sin y \sin (y + 60) = 0.$$

(16) Prove that $\dfrac{\sin 4A}{1 + \cos 2A} \cdot \dfrac{\cos 2A}{1 + \cos 4A} = \tan A$.

(17) Prove that $\dfrac{4 \tan A (1 - \tan^2 A)}{(1 + \tan^2 A)^2} = \sin 4A$.

(Additional elementary examples on the work of this chapter will be found on page 177.)

EXAMPLES XV

(1) Prove that
$$\sec^2 A + \sec A \tan A = \frac{1}{1 - \sin A}.$$

(2) Find the value of θ, less than 90°, which satisfies the equation $\sec \theta + \tan \theta = 2$.

(3) State the values of—
$\sin^{-1} 0{\cdot}8$, $\cos^{-1} 0{\cdot}8$, arc tan $1{\cdot}2$, $(\sin 30°)^{-1}$, $\sin^2 40°$, $(\cos 60°)^{-2}$.

(4) Find the values of—
$$\sin 240°,\ \cos 240°,\ \tan 240°$$
and, using these values show that
$$(\sin 240°)^2 + (\cos 240°)^2 = 1$$
and that
$$\frac{\sin 240°}{\cos 240°} = \tan 240°.$$

(5) Express the following in terms of the trigonometrical ratios of the angle A—
$\sin (90 + A)$, $\cos (180 - A)$, $\tan (270 - A)$, $\text{cosec} (270 + A)$, $\sec (360 - A)$.

(6) Find the values of—

$$\cos\frac{5}{3}\pi,\ \sin\frac{7}{2}\pi,\ \sec\frac{10}{3}\pi,\ \operatorname{cosec}\frac{11}{4}\pi,\ \cot\left(-\frac{5}{2}\pi\right).$$

(7) In a triangle ABC side $AB = 5$ in., angle $ACB = 35°$, and angle $BAC = 50°$. Solve the triangle.

(8) A ship is steaming N. 70° E. at 20 m.p.h., and the bearing of a lighthouse from it is N. 37° E. Twenty minutes later the bearing of the lighthouse is N. 29° W. Calculate the distances of the ship from the lighthouse when the two bearings were taken.

(9) From a point due south of a mountain the angle of elevation of its summit is 15° 32′. From another point, also due south, but 2000 yd. nearer than the first point, the angle of elevation of the summit is 37° 20′, the two points of observation being at the same level. Calculate the height of the mountain.

(10) Find the values of $\cos 3A$ and $\tan 3A$ in terms of trigonometrical ratios of the angle A.

(11) (*a*) Show that the cosine of an angle is equal to the sine of the complement.

(*b*) Use the tables to show that

$$2 \sin 110° \cos 110° = \sin 220°.$$

(*c*) If L is any angle from 0° to 90°, both inclusive, find the greatest and least values of

$$6076{\cdot}8 - 31{\cdot}1 \cos 2L.$$ (*E.M.E.U.*)

(12) B is a point 4·2 miles south of a point A. The direction from A of a ship, P, is 16° south of east, and the direction of P from B is 30° east of north. Calculate—

(i) the distance of P from A,

(ii) the distance from A of the point on the straight line AB, from which the direction of P is due north-east. (*N.C.*)

(13) (*a*) Using the formula for $\sin (A + B)$ find the value of the following expression—

$$\sin \theta + \sin (\theta + 120°) + \sin (\theta + 240)°;$$

(*b*) Evaluate $\frac{r^2}{2}(\theta - \sin \theta)$, when $\theta = \frac{\pi}{3}$ radians, $r = 6$. (*U.L.C.I.*)

(14) (*a*) If $\sec A = \frac{13}{5}$ find the value of

$$\frac{2 \sin A - 3 \cos A}{4 \sin A - 9 \cos A}.$$

(*b*) Derive an expression for the area of a sector of a circle of radius r, the angle of the sector being θ radians. (*U.E.I.*)

(15) (*a*) Prove the identity $\sin^2 A + \cos^2 A = 1$ when A is an acute angle.

(*b*) Verify that the above identity is correct when $A = 45°, 150°, 240°, 420°$.

(*c*) Use this identity to solve the equation $8 \sin^2 \theta = 9 - 6 \cos \theta$. (*E.M.E.U.*)

(16) P and Q are points on a straight coast-line, Q being 5·3 miles east of P. A ship, starting from P, steams 4 miles in a direction $65\frac{1}{2}°$ north of east. Calculate—

(i) the distance the ship now is from the coast-line;

(ii) the ship's bearing from Q. (*N.C.*)

(17) In a triangle ABC the angle $CAB = 44°$ and the angle ABC is obtuse. If $AC = 5$ in. and $BC = 3{\cdot}5$ in., calculate, to the nearest degree, the angles ACB and ABC and the length of AB. (*U.L.C.I.*)

(18) In a machine for testing the hardness of metals a steel ball of diameter 10 mm was pressed on the specimen, and the diameter of the impression was found to be 3·75 mm. Calculate to three significant figures the depth of the impression. (U.E.I.)

(19) (a) Express 160°, 170°, 180°, 190° in radians, and give their tangents.

(b) Find the value of

$$2c\cos\theta + (\pi + 2\theta)(a + b)$$

where $\sin\theta = \dfrac{a+b}{c}$

and $(\pi + 2\theta)$ is an angle in radians

and $a = 2{\cdot}5$, $b = 1{\cdot}5$, $c = 20$. (E.M.E.U.)

(20) Prove that if ABC is an acute-angled triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

At 2 p.m. a ship sailing due west at 18 m.p.h. sighted a lighthouse bearing 56° west of north. At 2.30 the bearing was 21° west of north. At what distance was the lighthouse when first sighted, and at what time will the distance from the ship to the lighthouse be least? (E.M.E.U.)

(21) (a) Show that the area of a triangle ABC can be expressed in the form $\frac{1}{2} AB \times AC \times \sin BAC$.

Determine the area when $AB = 8$ in. $AC = 5$ in., and the angle $BAC = 100°$.

(b) The circular steam outlet pipe from a safety valve is 4 in. external diameter; it passes vertically through a roof inclined at 60° to the axis of the pipe. State the shape of the aperture in the roof and determine its area. (U.L.C.I.)

CHAPTER III

VECTOR QUANTITIES

BRIEFLY expressed, a *vector* quantity is one which has *direction* as well as *magnitude*. Force, velocity and acceleration are examples of vector quantities, since, in order to specify them completely, we must give their *directions* as well as their magnitudes. Thus, we speak of a velocity as 10 m.p.h. in (say) *a north-easterly direction.*

On the other hand, length and mass are *scalar* quantities in which no question of direction is involved.

Engineering students are very largely concerned with vector quantities, and it should be understood quite clearly that when such quantities are to be added together, or subtracted from one another, the methods to be used are different from those which apply to the addition and subtraction of scalar quantities.

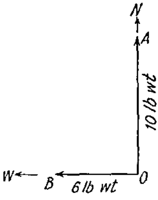
For example, if we add to a mass of 10 lb. another mass of 6 lb. we have, as a result, a mass of 16 lb. In this case the addition is by simple arithmetic. If, however, a force of 10 lb. weight acts on a body in a northerly direction, and we introduce a force of 6 lb. weight in a westerly direction, the total effect is *not* that due to 16 lb. weight, nor is its direction either north or west. Actually, the *resultant* force is one between 10 lb. and 16 lb. weight, and it acts in a direction somewhere between north and west.

To calculate the resultant force exactly we must apply the rules for the addition of vector quantities.

Graphical Representation of Vector Quantities. Instead of writing "a force of 10 lb. weight in a northerly direction," "a force of 6 lb. weight in a westerly direction," and so on, a graphical method of representation has been invented which not only simplifies the method of specifying a particular vector quantity but also assists one in solving problems which involve such quantities.

In the graphical representation these quantities are represented by lines whose *lengths* represent—to a certain suitably chosen scale—the *magnitudes* of the quantities and whose *directions* represent the directions of the vector quantities.

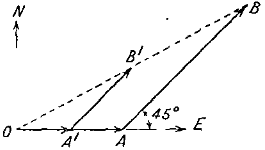
Thus, referring to Fig. 17,† the line OA represents a force of 10 lb. weight in a northerly direction, and line OB represents a force of 6 lb. weight in a westerly direction. It should be noted that the *scale* to which the vectors are drawn is stated below the drawing. This should *always* be done, and the scale should be chosen to give a conveniently large drawing. The larger the drawing the less will be the effect of errors in the measurement of lengths on it upon the accuracy of the result. The significance of this statement will, perhaps, be better appreciated when the next few pages have been read.



Scale. 1cm = 2 lb wt

FIG. 17

Graphical Determination of the Resultant of Two Vector Quantities. Suppose that a ship is being driven by its engines in a north-easterly direction at 10 m.p.h. and that an ocean current is carrying it in an easterly direction at 6 m.p.h. Under these



Scale 1 inch = 4 miles

FIG. 18

conditions the ship has two independent velocities at one and the same time. What will be its total, or resultant, velocity?

Consider its motion during one hour. In this time it will move 6 miles due east as represented by line OA in Fig. 18 and, also, it will move 10 miles north-east as represented by line AB. Hence, its total motion in one hour will be from

† The scales shown in this and succeeding figures indicate convenient ones for actual drawing. They do not refer to the figures as printed in the book, which are reduced in size.

O to B. Now if we measure the length OB and convert this length to miles (according to the scale used in the drawing) this will give us the distance moved by the boat in one hour.

If we had considered the movement in half an hour only we should have drawn OA' 3 miles and $A'B'$ 5 miles. Then OB' would have given the total motion of the ship in half an hour. From the geometrical properties of "similar" triangles (discussed in the next chapter) it follows that the point B' will lie on line OB and, also, that $OB' = \frac{1}{2} OB$. Hence the distance OB', travelled in half an hour, is half that travelled in one hour and *is in the same direction OB.*

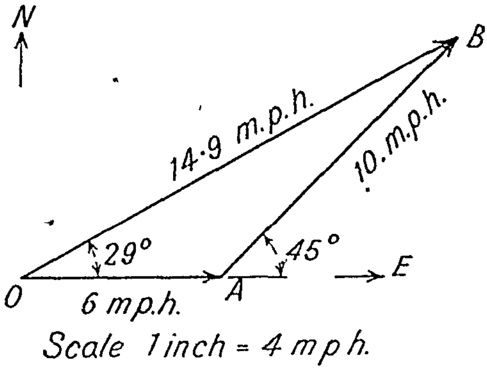


FIG. 19

In Fig. 18 the lines represent distances in miles, but from what has been said it is obviously unnecessary to consider specific times such as one hour or half an hour. We can draw a vector diagram of velocities as in Fig. 19, in which line OA represents 6 m.p.h. east and AB 10 m.p.h. north-east.

The resultant velocity is, from measurement, 14·9 m.p.h. in a direction N. 61° E. (or 29° North of East).

The triangle in Fig. 19 is called the *triangle of velocities.* In just the same way we have a *triangle of forces.* Thus if a body is acted on by a force of 6 lb. weight east and by another force 10 lb. weight north-east the resultant force, as obtained from a triangle of *forces* exactly similar to the triangle of Fig. 19, would obviously be 14·9 lb. weight in direction N. 61° E.

Note that arrows should always be placed on these vectors to indicate direction O to A (say) as distinct from direction A to O.

The Parallelogram Law. An alternative method of finding the resultant, or *vector sum,* of two vector quantities is to draw

a parallelogram as in Fig. 20, in which OA and OC represent the two vector quantities to a suitable scale and the diagonal OB represents their resultant, or vector sum, both in magnitude (to the same scale) and in direction.

It is obvious that this method is exactly the same in principle as the triangle method, since, in fact, the triangle OAB on this figure corresponds exactly to the triangles OAB of Figs. 18 and 19. (This follows since the opposite sides of a parallelogram are equal, so that $OC = AB$.)

In certain cases it is somewhat more convenient to use the

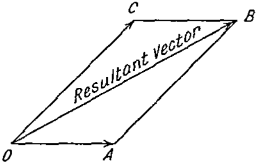


FIG. 20

parallelogram rather than the triangle, since it avoids any confusion regarding the direction of the arrow on the resultant vector. For this reason the parallelogram method will be used in the succeeding work rather than the triangle method.

EXAMPLES XVI

(1) Two forces, whose magnitudes are 7-lb. weight and 8-lb. weight respectively, act on a body, the angle between their directions being 120°. Find the resultant force on the body and the angle which the direction of this resultant makes with the 7-lb. force.

(2) If P and Q are two forces acting on a body and α is the angle between their directions, find the resultant force R in each of the following cases—

(i)	$P = 5$ oz.;	$Q = 9$ oz.;	$\alpha = 54°$.
(ii)	$P = 8$ lb.;	$Q = 12$ lb.;	$\alpha = 110°$.
(iii)	$P = 3$ tons;	$Q = 4$ tons;	$\alpha = 90°$.

In the last case check your answer by calculation (use Pythagoras's Theorem).

(3) A man wishes to cross a river in a rowing-boat so as to arrive at a point on the opposite bank exactly opposite to the point from which he started. If he can row at 4 m.p.h. and if the velocity of the stream is 3 m.p.h. in what direction must he row? How long will it take him to cross the river if it is 200 yd. wide?

(4) If the man referred to in Question 3 rowed in a direction perpendicular to the stream, how long would he take to cross and how far down-stream would he be carried before reaching the opposite bank?

Vector Subtraction. So far we have dealt only with the *addition* of two vector quantities. We must also consider *subtraction*, or the *difference* between two vectors, as well as their sum.

Before we can carry out subtraction we must introduce conventional positive and negative signs as applied to the vector quantities, just as we did in graphical work in preceding chapters. Thus, if we call a velocity or force in a direction west to east positive, then the reversal of this—namely east to west—is negative. Again, if a force P acts in a direction N. 30° E. the reversal of this, acting in the direction S. 30° W., is the force $-P$ as illustrated by Fig. 21.

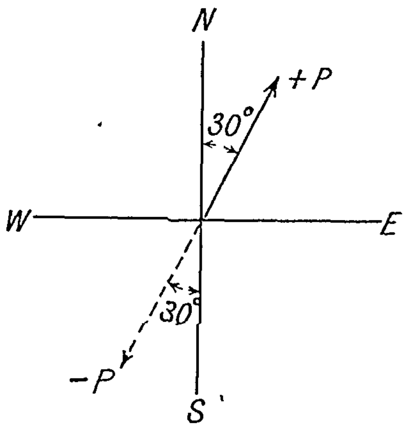


FIG. 21

Now, when we were adding two vector quantities we found that we could *not* express the sum of two vectors A and B arithmetically as $A + B$, since their sum was not simply $A + B$ but was obtained by a parallelogram.

We write the vector sum as

$$[A] + [B],$$

the square brackets indicating vector quantities and, therefore, that the summation is a vector summation and not merely an arithmetic sum.

Similarly we write a difference of two vectors A and B as

$$[A] - [B].$$

This can be re-written as

$$[A] + [-B]$$

and this indicates a vector summation of vector A and *the reversal of vector B*, i.e. of vector A and $-$ (vector B).

Referring to Fig. 22 the lines OA and OB represent vectors A and B respectively; OB' represents $-$(vector B); and hence the diagonal OC of the parallelogram whose sides are

OA and OB' represents the vector difference of vector A and vector B (i.e. $[OC] = [OA] + [-OB] = [OA] - [OB]$).

EXAMPLE 1. A ship is steaming due west with a velocity

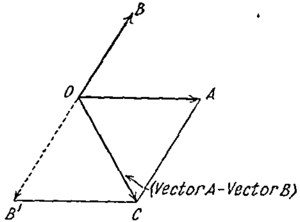


FIG. 22

20 m.p.h. Some time later it is moving S.W. at 15 m.p.h. Find its change in velocity in both magnitude and direction.

The change in velocity can be expressed as

(Final velocity) – (Initial velocity),

but the subtraction must, of course, be vector subtraction and not arithmetical.

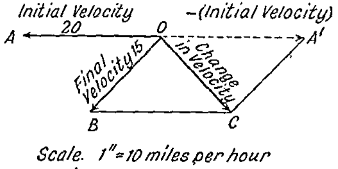


FIG. 23

In Fig. 23 vector OA represents the initial velocity of 20 m.p.h. west and OB the final velocity of 15 m.p.h. south-west. Vector OA' represents – (Initial velocity), and this,

when added vectorially to the final velocity, gives (Final velocity) − (Initial velocity) = Change in velocity.

In the figure, therefore, the change in velocity is represented by vector OC and the change is, by measurement, 14·4 m.p.h. in direction S. $42\frac{1}{2}°$ E.

EXAMPLE 2. Forces of 50 lb. and 40 lb. pull on a body at a point A, the angle between their directions being 60°. Find the resultant pull. If the 40 lb. force is converted to a push

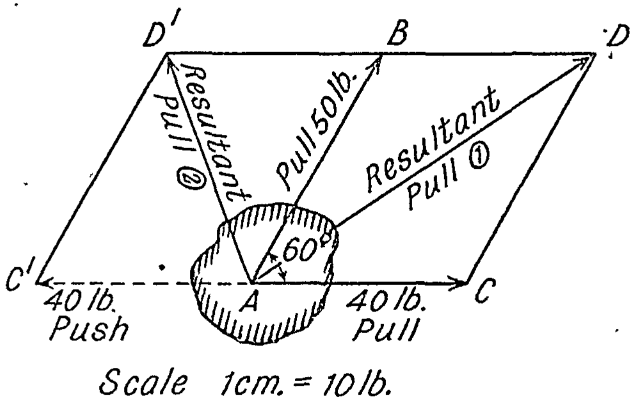


FIG. 24

instead of a pull find the magnitude and direction of the new resultant force.

Referring to Fig. 24 the vectors AB and AC represent, to scale, the pulls of 50 lb. and 40 lb. acting at A. The angle $BAC = 60°$. The diagonal AD of the parallelogram $ABDC$ gives, to the same scale, the first resultant pull.

By measurement this is 78 lb. in a direction making an angle $33\frac{1}{2}°$ with direction AC.

Regarding the second resultant, a *push* of 40 lb. is obviously the same in effect as a pull from right to left—i.e. in the opposite direction to the original 40 lb. pull.

Thus, to find this second resultant we draw AC reversed as shown and complete the parallelogram $ABD'C'$, the new resultant being given, to scale, by the diagonal AD'. By measurement, this resultant is 45·7 lb. in a direction making an angle $108\frac{1}{2}°$ with the direction AC.

Equilibrium and Equilibrating Force. A force may be defined as that which tends to move a body which is at rest, or to alter the motion of a body which is already moving. Thus if

a single force acts on a body the body will move. From the foregoing paragraphs it can be understood that if two forces are acting on the body these two forces will, in general, have a resultant which will be different both in magnitude and direction from either of them. The body will, however, still be caused to move, since the two forces can be replaced by a single force—namely, their *resultant*. But if one force is *exactly equal and opposite in direction* to the other the resultant will be zero, and the body will not move. Again, if the two forces are not equal and opposite and a third force, *equal in*

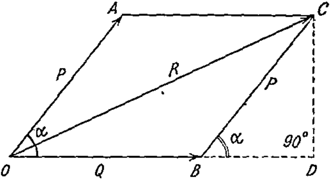


FIG. 25

magnitude but exactly opposite in direction to their resultant is applied to the body, the total resultant of the three forces will be zero, and the body will remain at rest. Under these conditions the body is said to be *in equilibrium* under the action of the three forces, and the third force—equal and opposite to the resultant of the other two—is called the *equilibrating force*. The same applies if there are more than two forces acting on the body, provided that the force which is added to produce equilibrium is always equal and opposite to the resultant of the forces which are already acting on the body.

Calculation of the Resultant of two Vectors. *If two vectors of magnitudes P and Q act at a point, the angle between their directions being a, then their resultant R is given by the formula*

$$R^2 = P^2 + Q^2 + 2PQ \cos a.$$

This formula—which should be carefully memorized †—can be proved quite easily by reference to Fig. 25.

† Note the + sign preceding the term $2PQ \cos a$. In Book III we shall deal with a very similar formula in Trigonometry in which this sign is *minus* instead of *plus* as here.

Proof. Sides OA and OB of the parallelogram $OACB$ represent to scale the vectors P and Q, the angle AOB between them being α. The diagonal OC represents their resultant R.

In the figure the line OB is produced to meet a perpendicular from C in the point D.

Then $CDB = 90°$ and, since CB is parallel to AO, $C\hat{B}D = \alpha$. Also, $CB = AO = P$.

In the right-angled triangle COD

$$\begin{aligned} CO^2 &= CD^2 + OD^2 \qquad \text{(from Pythagoras's Theorem)} \\ &= CD^2 + (OB + BD)^2. \end{aligned}$$

Now, from triangle CBD, $CD = CB \sin \alpha$
$= P \sin \alpha$

and, also, $BD = CB \cos \alpha$
$= P \cos \alpha.$

Again $CO = R$ and $OB = Q$.

$$\begin{aligned} \therefore \quad R^2 &= (P \sin \alpha)^2 + (Q + P \cos \alpha)^2 \\ &= P^2 \sin^2 \alpha + Q^2 + 2PQ \cos \alpha + P^2 \cos^2\alpha \\ &= P^2 (\sin^2 \alpha + \cos^2 \alpha) + Q^2 + 2PQ \cos \alpha. \end{aligned}$$

But, $\sin^2 \alpha + \cos^2 \alpha = 1$.

$$\therefore \quad R^2 = P^2 + Q^2 + 2PQ \cos \alpha.$$

To illustrate the use of this formula consider the example which was solved graphically in the preceding paragraph. Then, by substitution, the first resultant is given by

$$\begin{aligned} R^2 &= 50^2 + 40^2 + 2 \times 50 \times 40 \times \cos 60° \\ &= 2500 + 1600 + 4000 \times \tfrac{1}{2} \\ &= 6100 \\ \therefore \quad R &= \sqrt{6100} \\ &= 78{\cdot}1 \text{ lb.} \end{aligned}$$

In the second case the angle between the two forces is not 60° but 120°, whose cosine is $-\frac{1}{2}$.

$$\begin{aligned} \therefore \quad (R')^2 &= 50^2 + 40^2 - 2 \times 50 \times 40 \times \tfrac{1}{2} \\ &= 2100 \\ \therefore \quad R' &= \sqrt{2100} \\ &= 45{\cdot}8 \text{ lb.} \end{aligned}$$

It can be seen that these results compare quite well with those obtained graphically.

If it is desired to calculate the directions as well as the magnitudes of the resultants this can be done by using the sine formula dealt with in Chapter II.

Thus, referring to Fig. 24,

$$\frac{DC}{\sin D\hat{A}C} = \frac{AD}{\sin A\hat{C}D}.$$

Now $DC = 50$ and $AD = 78{\cdot}1$ (to scale).

Also, $A\hat{C}D = 120°$ since $B\hat{A}C = 60°$ and BA and DC are parallel

$$\therefore \qquad \frac{50}{\sin D\hat{A}C} = \frac{78{\cdot}1}{\sin 120°}$$

$$\text{or} \qquad \sin D\hat{A}C = \frac{50 \sin 120°}{78{\cdot}1}$$

$$= \frac{50 \times 0{\cdot}8660}{78{\cdot}1} = 0{\cdot}5544$$

$$\therefore \qquad D\hat{A}C = 33° \ 40'.$$

This means that the resultant—represented by AD—makes an angle of $33° \ 40'$ with direction AC.

The direction of AD' can be calculated in a similar way.

Acceleration. By *acceleration* we mean *rate of change of velocity*. Consider a body moving in a straight line with a velocity of 2 ft. per sec. If after two seconds its velocity is 5 ft. per sec. *in the same direction* the change in velocity has been 3 ft. per sec. in two seconds. Its acceleration may be expressed as *3/2 ft. per sec. per sec.* Strictly, this is the *average acceleration*; by dividing the total change in velocity by the total time and stating the acceleration as 3/2 we are assuming that the acceleration has been uniform during the two seconds.

Instead of taking *1 ft. per sec. per sec.* as our unit of acceleration, as in the above, we might, of course, take such units as *1 cm. per sec. per sec.*, *1 m.p.h. per min.*, or, in the case of angular velocity, *1 radian per sec. per sec.* In all cases the unit must express rate of change of velocity.

Acceleration is not always in the line of motion; the velocity of a body might change in five seconds from 10 ft. per sec. due east to 20 ft. per sec. north-east. In such a case the change in velocity is the *vector difference* between these two velocities, and not the arithmetic difference $(20 - 10)$. It follows, therefore, that the acceleration has a direction, as well as a magnitude, both of which must be found from a vector diagram.

Fig. 26 refers to this case.

Vectors OA and OB represent the initial and final velocities respectively. The change in velocity is the vector difference $[OB] - [OA]$, i.e. the vector sum of OB and OC given by vector OD.

By measurement, therefore, the change in velocity in five seconds is 14·7 ft. per sec. in direction N. 16° E., so that the

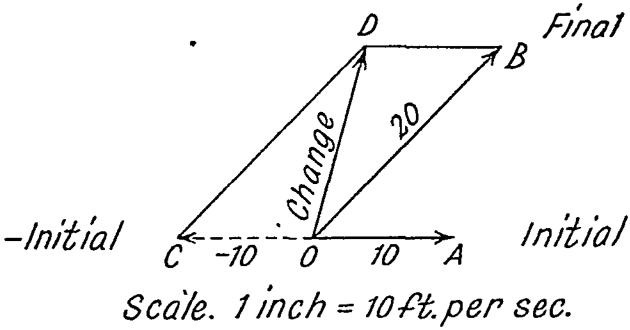


FIG. 26

acceleration—assumed uniform—is $14{\cdot}7/5 = 2{\cdot}94$ ft. per sec. per sec. in direction N. 16° E.

Momentum. The *momentum* of a body is the product of its mass and its velocity, or

$$\text{Momentum} = \text{Mass} \times \text{Velocity}.$$

Now since mass is a scalar, not a vector, quantity, having magnitude, but no direction, we are merely multiplying a vector (velocity) by a number (mass), and thus momentum is a vector quantity. It follows that the rules which have been stated regarding change of velocity and rate of change of velocity apply in exactly the same way to momentum. Thus, change in momentum = mass × change in velocity, or rate of change of momentum = mass × rate of change of velocity, the change in velocity being determined vectorially as previously explained.

EXAMPLES XVII

(1) Find, graphically, the vector *difference* of two vectors of 7 and 8 units acting at an angle of 60° apart.

(2) A body is acted on by two forces—a pull of 20 lb. horizontally and a push of 30 lb. vertically downwards. Find the magnitude and direction of a third force which, if caused to act on the body, would prevent its movement under the action of these two forces.

(3) If P and Q are two forces acting at a point, the angle between them being a and their resultant being a force R, calculate—

(i) R when $P = 5$ lb., $Q = 6$ lb., and $a = 50°$.
(ii) a when $P = 8$ oz., $Q = 9$ oz., and $R = 15$ oz.
(iii) P when $Q = 3$ tons, $R = 5$ tons, and $a = 90°$.
(iv) R when $P = 120$ grams, $Q = 180$ grams, and $a = 120°$.

(4) A ship is being driven by its engines at 10 m.p.h due east. It is being carried by a current whose rate is 8 m p.h. and its resultant velocity is in a direction N. 50° E. Calculate the direction of the current and the magnitude of the resultant velocity. (Use the sine formula to carry out the calculation.) Check your answers graphically.

(5) A body is moving at 20 ft. per sec. and five seconds later it is moving still at 20 ft. per sec, but in a direction making an angle of 50° with the first direction. Find the magnitude and direction of the acceleration, supposed constant.

(6) A body which is moving at 15 m.p h. due north has an acceleration of 2 ft. per sec per sec in a N.W. direction. Find the magnitude and direction of its velocity after eight seconds.

Relative Velocity. When we speak of the velocity of a point A *relative to a point* B (both of which points are, in the general case, assumed to be moving) we mean "*that velocity which A must have if it alone moves and B remains stationary, in order that the displacement between the two points in a given time may be the same in magnitude and direction as when both the points were moving.*" Alternatively, the velocity of A relative to B is the one which it appears to have when looked at from the moving point B.

To take an example, suppose a train A is moving along a railway line with a velocity of 30 m.p.h., and that another train B is following it at 20 m.p.h. Then the velocity of train A relative to train B is 10 m.p.h., since A obviously gains on B at this rate. In other words, if B remained stationary and A proceeded at 10 m.p.h the two trains would be the same distance apart at the end of any given time as they would be if they moved at 30 and 20 m.p.h. respectively.

Strictly speaking all velocity is relative; when we speak of a velocity as 20 m.p.h. we mean—although we do not often say so, since it is commonly understood—20 m.p.h. *relative to the surface of the earth.* But, as we all know, the earth is itself moving, so that the actual velocity of the train in space is something very different from 20 m.p.h. For most purposes, however, we assume the earth's surface to be stationary, and call what is actually the "velocity of the train relative to the earth" the "velocity" of the train.

Now, in the above example of the two trains, we obtained the velocity of A relative to B by subtracting B's velocity from A's velocity. We can, therefore, express this in the form of an equation as follows—

Velocity of A relative to B

$$= A\text{'s velocity} - B\text{'s velocity}.$$

It is important to note that this equation holds good no matter whether the two velocities are in the same direction or not, provided the rules for vector subtraction, as explained in the preceding paragraphs, are applied in all cases.

EXAMPLE 1. Two trains, A and B, are moving *in opposite directions* along a railway line with velocities of 30 m.p.h. and 20 m.p.h. respectively. Find the velocity of A relative to B.

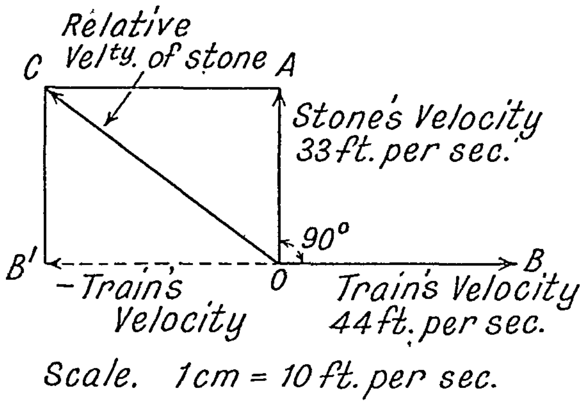


FIG. 27

In this case, from the vector point of view, the velocity of B is -20 m.p.h. Hence

Velocity of A relative to B = A's velocity $-$ B's velocity.

$$= 30 - (-20)$$
$$= 50 \text{ m.p.h.}$$

Note that a vector diagram is unnecessary for the subtraction in this case, since the two velocities are in one straight line, although they are opposite in direction.

A little consideration will show that this answer is correct, since, if the trains start from the same place and move in opposite directions with velocities of 30 m.p.h. and 20 m.p.h., they will obviously be 50 miles apart at the end of one hour.

EXAMPLE 2. A train is travelling at 30 m.p.h. and a stone is thrown at it with a velocity of 33 ft. per sec. in a direction perpendicular to that of the train. Find the magnitude and direction of the velocity with which the stone appears, to a person on board the train, to be approaching the train—i.e. find the velocity of the stone relative to the train.

Now the velocity of the stone relative to the train is

Stone's velocity − Train's velocity.

Since these two velocities are not in the same direction the subtraction must be carried out vectorially, as previously explained. This is done in Fig. 27, in which OA and OB represent, to scale, the velocities of the stone and train respectively in feet per second. (Note that OB is made to represent 44 ft. per sec. since this is the equivalent of 30 m.p.h.)† OB' represents the reversal or negative of the train's velocity, and when this vector is added by parallelogram to the vector OA the resultant velocity, represented by OC, gives the velocity of the stone relative to the train.

By measurement this velocity is found to be 55 ft. per sec. in a direction making an angle of 126° 52′ with the direction OB of the train.

EXAMPLES XVIII

(1) Two ships, A and B, are sailing with velocities 15 m.p.h. N.W. and 20 m.p.h. N. 30° E. respectively. Find the velocity of A relative to B.

(2) To a person on board a ship which is steaming at 15 m.p.h. due north another ship appears to be steaming north-east. If the actual speed of the second ship is 18 m.p.h. find its actual direction of movement.

(3) Two motor-cars start together from a cross-roads and move along two different roads with steady speeds of 20 m.p.h. and 30 m.p.h. respectively. At what angle are the roads inclined to one another if the actual distance between the two cars increases at the rate of 28 m.p.h.?

(4) Two aeroplanes, A and B, are flying at the same height and their velocities are, respectively, 250 m.p.h. due north and 180 m.p.h. in direction S. 47° W. With what velocity and in what direction does A appear to be moving as viewed by a person on board aeroplane B?

(5) To a person travelling at 20 m.p.h. in a train which runs along the coast in direction south-east a ship appears to be sailing in a direction perpendicular to the coast (i.e. north-east). If the ship is actually sailing at 25 m.p.h., in what direction is it sailing?

(6) A ship is sailing due west at 20 m.p.h. and a second ship, which is lying 15 miles due south of the first, wishes to overtake it. If the second ship can sail at 28 m.p.h., in what direction must it sail and how long will it take to overtake the first?

The Polygon of Forces. When more than two forces are acting at a point obviously we cannot use the triangle or parallelogram methods of graphical solution to find their resultant.

In Fig. 28 (a) there are four forces acting at a point O, these

† It is convenient to remember that 60 m.p.h. = 88 ft. per sec.

forces being *co-planar*—i.e. lying in one plane. In Fig. 28 (*b*) these forces are represented in magnitude and direction diagrammatically by the four lines *oa*, *ab*, *bc*, and *cd*. It will be noticed that these four lines do not form a closed figure (polygon). Now, it can be shown experimentally that if such a

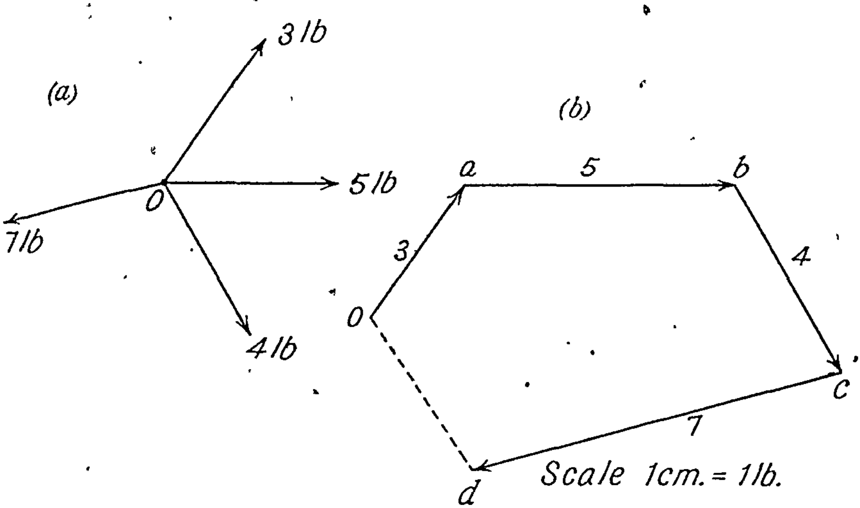


FIG. 28

diagram of forces as that of Fig. 28 (*b*) forms a *closed* polygon the forces which its sides represent are in equilibrium. It follows, therefore, that the dotted line *do* which closes the polygon represents, in magnitude and direction, a fifth force, which, if applied together with the four shown in Fig. 28 (*a*) and in the same plane, would produce equilibrium, i.e. the line *do* represents the *equilibrating force*. Now it was shown on page 68 that the equilibrating force is always equal in magnitude and exactly opposite in direction to the resultant. Hence the line *do* represents the magnitude of the resultant of the forces shown in Fig. 28 (*a*) and the *direction* of the resultant is *od*—i.e. opposite to the direction *do* of the equilibrating force.

By measurement of the line *od* in Fig. 28 (*b*) we find, therefore, that the resultant of the four forces is 3·4 lb., and its direction is, of course, *od*.

Another method of finding the resultant of a number of forces acting at a point will be given later in the chapter.

EXAMPLES XIX

(1) Find the magnitude and direction of the resultant of four co-planar forces of 2 tons, 2·5 tons, 3 tons and 4 tons, the angles between them being—

Between 2 and 2·5	. 50°
Between 2·5 and 3	. 70°
Between 3 and 4	. 120°
Between 4 and 2	. 120°

(2) Five forces of 7 lb., 5 lb., 4 lb., 8 lb. and 6 lb. act in one plane at a point in directions north, north-west, S. 70° E., S. 35° W., N. 40° E. respectively. Find the magnitude and direction of their resultant.

(3) Four co planar forces A, B, C and D act at a point. Their values are: A = 50 lb. due north, B = 80 lb. due west, C = 70 lb. N. 75° E. and D = 40 lb. S. 10° E. Are those forces in equilibrium? If not find, by the polygon method, a fifth force which, together with the existing four, would produce equilibrium

(4) Referring to Question 3, find, by parallelogram, the resultant (S) of A and B, then the resultant (T) of C and D. Draw another parallelogram with these resultants S and T as its sides and show that this method gives the same result as the polygon method used in Question 3.

Components and Resolution of Forces. Earlier in the chapter we have discussed the resultant of two forces acting at a

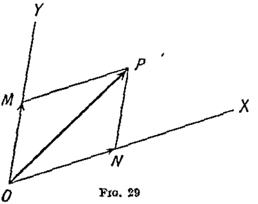


FIG. 29

point, the resultant being a single force which has the same effect upon the point on which it acts as do the two forces acting together. In the same way we can replace a single force by two forces acting at the same point, these two forces being in any desired directions. Such forces are called the *components* of the single force in the directions chosen.

Thus, in Fig. 29 a single force P acting along OP is split up into two components in directions OX and OY. This is done graphically by drawing two lines from P parallel to OX and OY respectively. Then ON represents the component of P in direction OX, and OM the component in direction OY, to the

same scale as that to which the line OP—representing force P—is drawn.

It is obvious that forces represented by ON and OM are together equivalent to the single force P, since these lines are the sides of a parallelogram of which OP is the diagonal, i.e. P is the resultant of forces represented by ON and OM.

The above process is spoken of as the *resolution* of the force P into its two components in directions OX and OY.

A special case of such resolution occurs when the two directions OX and OY are at right angles to one another. The two components are then called the *resolved parts* or *rectangular components* of the force P in the two directions.

Referring to Fig. 30, the forces represented by ON and OM are the resolved parts of the force represented by OP in the directions OX and OY respectively. The lines PN and PM are, of course, perpendicular to OX and OY respectively.

From the triangle PNO it follows that the resolved part of the force P in direction OX (i.e. the total effect of P in direction OX) is $OP \cos \theta$, where θ is the angle between OP and direction OX. Similarly $OM = OP \cos (90 - \theta) = OP \sin \theta$. Hence, if we substitute the magnitude P of the force for the length OP, we can say that—

Resolved part of P in direction

$$OX = P \cos \theta$$

and, resolved part of P in direction

$$OY = P \cos (90 - \theta).$$

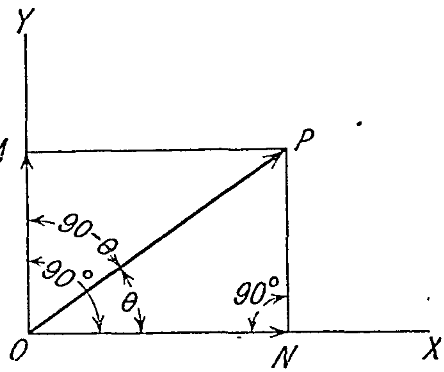


FIG. 30

In general *the resolved part (or total effect) of a force P in a direction making an angle θ with that of the force is $P \cos \theta$.*

It follows, also, from this statement that *the resolved part, or effect, of a force P in a direction making an angle of 90° with its own direction is $P \cos 90° = 0$.*

EXAMPLE. In Fig. 31 A is a pin joint and is in equilibrium. It is supported by a smooth horizontal surface, and the two members meeting at the joint are inclined at 60° to one

another as shown. If there is a downward thrust of 50 lb. in the inclined member find the force in the horizontal member and the pressure on the support.

Now, we can split up the 50-lb. force into two components or resolved parts, one vertically downwards and the other horizontally right to left. This splitting up is shown in the figure (not to scale), the resolved parts being P and Q.

Now, from what has been said $P = 50 \cos 60°$ and

$$Q = 50 \cos 30°, \text{ i.e. } P = 25 \text{ lb. and } Q = 43{\cdot}3 \text{ lb.}$$

Since the support is smooth it cannot exert any horizontal frictional force, so that the whole of the horizontal component P must be counterbalanced by a force of 25 lb. left to right (i.e. away from the joint) in the horizontal member.

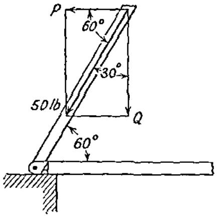


FIG. 31

This force in the horizontal member has no effect vertically (i.e. in a direction perpendicular to itself), and thus the pressure on the support is $Q = 43{\cdot}3$ lb.

Resultant of a Number of Co-planar Forces by the Resolution Method. The resolution method can be applied to form a very useful means of calculating the resultant of a number of co-planar forces acting at a point. This calculation method provides an alternative to the graphical method of determining such a resultant by the polygon of forces, which has already been discussed.

Fig. 32 (*a*) shows five co-planar forces—P, Q, S, T and W acting at a point—whose resultant is to be found. Two axes XX' and YY' are taken, perpendicular to one another. It does not matter in what directions these two axes are drawn so long as they are taken perpendicular to one another. Positive (+) and negative (–) signs are placed on them just as in drawing graphs, and the resolved parts of all the five forces in the directions XX' and YY' are found. The resolution is illustrated by dotted lines in the figure, but will be done by calculation as below.

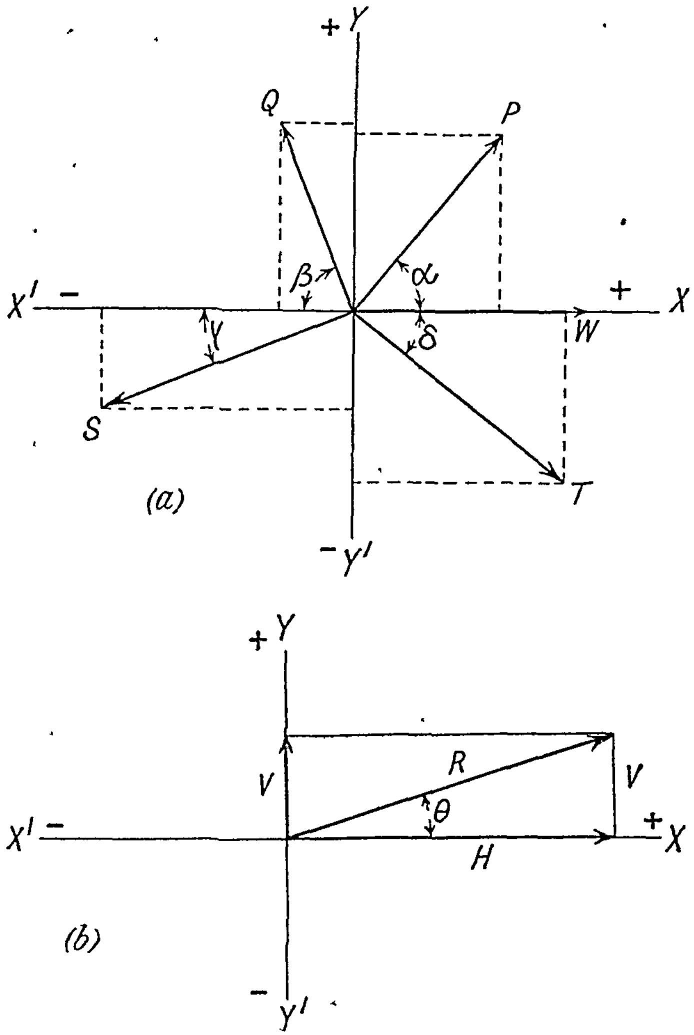


FIG. 32

Horizontal Components

Horizontal component of $W = + W \cos 0° = + W$

" " $P = + P \cos \alpha$

" " $T = + T \cos \delta$

" " $Q = - Q \cos \beta$

" " $S = - S \cos \gamma$.

$\therefore$ Total horizontal component of the five forces

$$H = W + P \cos \alpha + T \cos \delta - Q \cos \beta - S \cos \gamma.$$

Vertical Components

Vertical component of $W = W \cos 90 = 0$

" " $P = + P \cos (90 - \alpha)$

(since $90 - \alpha$ is the angle between P and YY').

$= + P \sin \alpha$

" " $Q = + Q \cos (90 - \beta) = + Q \sin \beta$

" " $S = - S \cos (90 - \gamma) = - S \sin \gamma$

" " $T = - T \cos (90 - \delta) = - T \sin \delta$.

$\therefore$ Total vertical component of the five forces

$$V = 0 + P \sin \alpha + Q \sin \beta - S \sin \gamma - T \sin \delta.$$

We have now reduced the five forces to two forces—H horizontally and V vertically. These are shown in Fig. 32 (*b*). It is now a simple matter to calculate the resultant of H and V. In the figure the parallelogram of forces for H and V is drawn and R is their resultant. Since this parallelogram is actually a rectangle we can use Pythagoras's Theorem and say that

$$R^2 = H^2 + V^2$$

or

$$R = \sqrt{H^2 + V^2}.$$

This gives the magnitude of the resultant of the five original forces, since we replaced the five by two equivalent forces—namely H and V—before using Pythagoras's Theorem.

The direction of the resultant R can be found from the fact that

$$\tan \theta = \frac{V}{H}, \text{ both } V \text{ and } H \text{ being known.}$$

EXAMPLES XX

(1) A canal barge is in the middle of a canal 50 ft. wide and a man on the bank, 60 ft. ahead of the barge, pulls with a force of 80 lb. on a rope attached to it. What is the force urging the boat forwards, and what force is tending to pull it towards the bank on which the man is standing?

(2) A crank 18 in. long is revolving at 300 r.p.m. What is the vertical component of the velocity of the end of the crank at the instant when the crank is inclined to the vertical at an angle of 50°?

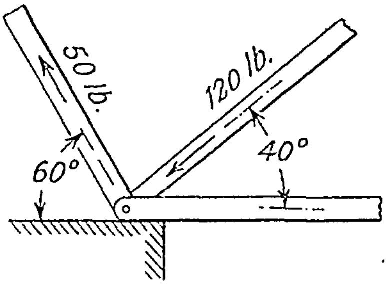


FIG. 33

(3) The pin joint shown in Fig. 33 rests on a smooth support and is in equilibrium. If the forces in the members inclined to the horizontal at angles of 40° and 60° are 120 lb. and 50 lb. respectively in the directions shown, calculate the force in the third member and the pressure on the support.

(4) Calculate the magnitude and direction of the resultant of five co-planar forces—

10 lb. due East; 8 lb. N. 20° W.; 7 lb. S. 80° W.; 6 lb. S. 30° E.; and 9 lb. N. 40° E.

(5) If, in Question 4, the second, third and fifth forces are reversed in direction what will then be their resultant?

Rotating Vector. Referring to Fig. 34 the line OP rotates in an anti-clockwise direction about the end O, starting from the position OX. In the figure OP has rotated through an angle θ from its zero position (i.e. OX). Now if a line is drawn through P parallel to OX this line meets the vertical line YY' in the point P'. To the right-hand side of the figure rectangular axes are drawn and the horizontal axis is marked out in a scale of degrees to represent values of θ—the angle moved through by the line OP as it rotates. At θ degrees along this scale an ordinate MN—of no special length—is set up. The line through P when produced to the right meets MN in the point Q.

Now, in the triangle POP',

$$\frac{OP'}{OP} = \cos P\hat{O}P' = \cos (90 - \theta) = \sin \theta,$$

$$\therefore \qquad OP' = OP \sin \theta.$$

Also, MQ is obviously equal to OP', so that MQ is also equal to $OP \sin \theta$.

If OP is of length R units, then, letting $OP' = y$, we have

$$y = R \sin \theta = MQ.$$

If the process of projection is continued for a number of values of the angle θ we shall obtain, to the right-hand side of the figure, the graph

$$y = R \sin \theta$$

as shown.

The line OP, of length R units, is called a *rotating vector* or, sometimes, a *radius vector.*

There are several quantities in engineering practice which vary according to a *sine law*, such as $y = R \sin \theta$. For example, the voltage generated in an alternating-current generator, or

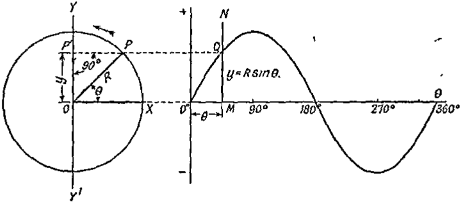


FIG. 34

alternator, obeys such a law, the angle θ in this case being the angle turned through by the rotating part of the alternator. Thus, if we let e be the voltage corresponding to any angle θ we have

$$e = E \sin \theta.$$

Now, we have seen already that the largest value of the sine of an angle is 1, i.e. when the angle is 90°. When $\theta = 90°$, therefore,

$$e = E \sin 90 = E.$$

E is thus the maximum value of the generated voltage. We may thus write for the voltage e corresponding to any angle θ

$$e = \text{maximum value of the voltage} \times \sin \theta.$$

We could use the rotating vector method to draw the graph of the voltage e by making the radius vector OP represent E

volts to scale (e.g. 1 cm. = 10 volts). The ordinates of the graph would then give the values of e to the same scale.

(Additional elementary examples on the work of this chapter will be found on page 178.)

EXAMPLES XXI

(1) Find graphically the magnitude and direction of the resultant of two forces of 5 tons and 8 tons pulling on a point, the angle between the forces being 55°.

(2) In a framework there is a pin joint at which three members A, B and C meet. The angle between A and B is 25°, and between B and C 35°. There is a thrust of 60 lb. in A and a pull of 150 lb. in C. If the joint rests in equilibrium on a smooth support, calculate the force in member B and the pressure on the support. The member C is horizontal.

(3) A man wishes to cross a river 150 yd. wide in which the current flows at $2\frac{1}{2}$ m.p.h. If he can row at 5 m.p.h. in what direction must he row in order to reach a point on the opposite bank directly opposite from where he started, and how long will it take him to cross?

(4) An aeroplane is ascending so that its altitude is increasing at the rate of 60 ft. per sec. The horizontal component of its velocity is 80 m.p.h. What is the magnitude and direction of its actual velocity?

(5) Find graphically the vector difference between a force of 9 lb. due north and a force of 6 lb. in a direction S. 65° E.

(6) The velocity of a body changes in three seconds from 30 m.p.h. N.W. to 20 m.p.h. S. 70° W. Find its acceleration.

(7) A thrust of 30 lb. and a pull of 50 lb. act on a point, the angle between them being 50°. Calculate the value of a third force which, acting on the point in the same plane as the other two, will keep the point in equilibrium.

(8) Calculate the magnitude and direction of the resultant of two forces of 2 tons and 3 tons acting at a point, the angle between them being 60°.

(9) Two bodies A and B are moving as follows: A moves at 30 ft. per sec. due east; B moves at 40 ft. per sec. N.W. What is the velocity of A relative to B?

If the bodies start from the same point together how far apart will they be after 5 sec.?

(10) Two trains, each 200 yd. long, are moving in the same direction along parallel lines. If one is travelling at 60 m.p.h. and the other at 40 m.p.h. how long will it take for the faster one to pass the slower, reckoning from the time at which the engine of the faster train overtakes the rear of the slower one.

(11) Four co-planar forces of 40 lb., 25 lb., 30 lb. and 50 lb. act at a point in directions N.W., S. 70° E., N. 35° E. and S. 40° W respectively. Find graphically the magnitude and direction of the fifth force which, acting together with them, will produce equilibrium.

(12) At a joint in a roof truss four members meet. Of these one is horizontal, one vertical, one is inclined to the vertical member at 25° and is on the same side of it as the horizontal member, while the fourth is inclined to the vertical member at an angle of 50°, and is on the opposite side of it from the other two members. Find the forces in the horizontal and vertical members if the member inclined to the vertical at 25° carries a thrust of 2 tons and the fourth member carries a pull of 3 tons, the joint being in equilibrium.

(13) A force of 10 lb. acts on a point. Find two forces which, acting in directions making angles of 50° and 25° on either side of the 10 lb. force, will have the same effect on the point as the latter.

(14) Five co-planar forces act at a point. These are as follows—

20 lb. due N.
35 lb. S. 53° E.
42 lb. N. 49° W.
25 lb. S 29° W.
16 lb. N. 20° E.

Calculate the magnitude and direction of their resultant.

(15) A certain quantity x varies according to the law

$$x = 5 \sin \theta.$$

Using the rotating vector method draw the graph of this quantity from $\theta = 0$ to $\theta = 360°$.

(16) A body is moving due north with momentum of 3000 units. Three seconds afterwards it is moving south-west with 4000 units of momentum.

Show by means of a Vector Diagram how to obtain the change of momentum. Calculate—

(i) The total change of momentum.
(ii) The average change of momentum per second. (*U.E.I.*)

(17) Four co-planar forces act on a pin in a structure. The forces are 3 tons acting east, 7 tons at 60° north of east, 5 tons at 20° north of west and 6 at 55° south of east.

All the forces are pulling away from the pin and may be considered as acting at a point. By a trigonometrical method find the magnitude and direction of a single force that would balance the above system of forces. (*U.E.I.*)

(18) Plot the curve $y = \sin \theta + 0 4 \sin 2\theta$, for values of θ between 0° and 180°. If you use your tables for the sines take values of θ by 20° increments.

A solution by rotating vectors will be accepted. (*U.L.C.I.*)

(19) Two forces of 3 5 tons and 6 7 tons act together in the same plane at a point. Both forces pull away from the point, and the angle between the directions of the forces is 75°.

Represent these forces on a diagram and complete the parallelogram of which they are adjacent sides.

The single force equivalent to the above two forces is represented by the diagonal of this parallelogram drawn through the meeting point of the forces. Calculate—

(i) The magnitude in tons of this single force.
(ii) The angle, to the nearest degree, which this single force makes with the line of action of the greater of the above forces.

N.B. Readings taken from a scale drawing will not be accepted as a solution. (*U.E.I.*)

(20) Steam enters the blades of a turbine with a velocity of 1590 ft. per sec. in a direction making an angle of 20° with the horizontal. During the passage through the blades the steam is turned through an angle of 107° and leaves the blades with a velocity of 465 ft. per sec. Draw to scale a diagram of velocities showing—

(i) The change in the velocity of the steam in passing through the blades;
(ii) The change of steam velocity in a horizontal direction.

Read off these changes of velocity in feet per second. (*U.E.I.*)

(21) A weight W lb. is to be suspended by two strings of lengths 7 ft. and 9 ft. respectively from two points 12 ft. apart in a horizontal beam. What is the maximum allowable value of W if the string used breaks under a tension of 50 lb.?

(22) At a certain time a ship which is sailing at 12 m.p.h. due south is

20 miles north-west of another ship which is sailing at 25 m.p.h. in a south-westerly direction. After how long will they be at their shortest distance apart and what will this distance be?

(23) Two rotating vectors are of lengths 5 cm. and 3 cm. respectively and rotate at the same speed. The shorter one has a fixed angular displacement of 60° behind the longer. Starting with the longer vector in the horizontal position, obtain, by projection, the corresponding sine curves.

(24) Four vectors A, B, C and D have magnitudes and directions as follows—

A . . .	10 units due S.
B . . .	8 units S. 40° W.
C . . .	7 units N. 58° E.
D . . .	6 units S. 25° E.

Using the method of the polygon of forces, find the value of—

$$(\text{Vector } A) - (\text{Vector } B) + (\text{Vector } C) - (\text{Vector } D),$$

expressing your answer in both magnitude and direction.

CHAPTER IV

GEOMETRY AND MENSURATION

Some Further Geometrical Facts. In Chapter V, Book I, we discussed a number of important geometrical facts and theorems relating to triangles and rectangles. We must now consider some geometrical properties of circles and of *similar* triangles.

Definitions.

A circle is a curve, lying in a plane, such that all straight lines drawn to it from a point within it, called its *centre*, are of equal length. One such line is called a *radius* (*pl. radii*) of the circle.

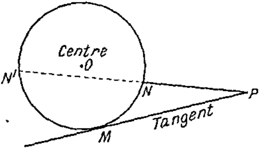


FIG. 35

An arc of a circle is any portion of its *circumference* or boundary.

Concentric circles are circles which have the same centre, but not the same radii.

A chord of a circle is a straight line joining any two points on the circumference, a *diameter* being any chord which passes through the centre. A diameter is obviously equal to twice the radius of the circle.

A tangent to a circle is a straight line which meets the circle in only one point, no matter how far the line is produced in either direction. Any other line drawn to meet a circle from a point outside it will, if produced, cut the circle in another point. This is illustrated in Fig. 35, in which two lines are drawn to meet a circle from a point P outside it. The line PM is a tangent, since it meets the circle only in the single point M

while the line PN will, if produced, cut the circle again in point N' as shown.

Theorems.

1. *If a chord of a circle does not pass through the centre, the straight line drawn from the centre to the mid-point of the chord is at right angles to the chord.*

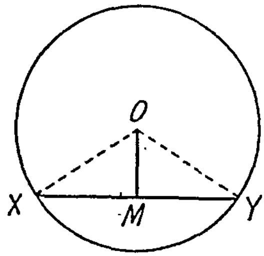


FIG. 36

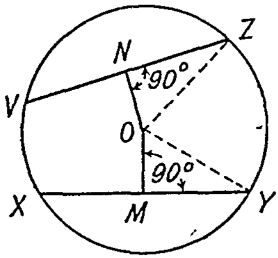


FIG. 37

In Fig. 36 XY is a chord and M its mid-point, a straight line OM being drawn from the centre O of the circle to this mid-point.

Now, if O is joined to X and Y, two triangles, XOM and YOM, are formed.

In these triangles

$OX = OY$ . . . (radii)

$XM = MY$ . . . (since M is the mid-point)

OM is common to both triangles.

$\therefore$ The triangles are congruent, or equal in all respects.

It follows that $X\hat{M}O = Y\hat{M}O = 90°$, so that OM is at right angles to XY.

Conversely, if a straight line is drawn from the centre of the circle perpendicular to a chord such as XY it bisects the chord.

2. *Two chords of a circle which are themselves equal are equidistant from the centre of the circle.*

In Fig. 37 XY and VZ are equal chords of the circle, and OM and ON are drawn from the centre perpendicular to these chords.

It is required to prove that these perpendicular distances are equal, i.e. that $OM = ON$.

If we join OY and OZ we form two triangles which are equal in all respects, since

$OY = OZ$. . . (radii)

$MY = NZ$ . . (since they are the halves of equal chords)

$O\hat{M}Y = O\hat{N}Z = 90°$.

Therefore the triangles are equal in all respects and

$$OM = ON.$$

Conversely, two chords which are equidistant from the centre are equal.

3. *The radius of a circle, drawn from the centre to the point of contact of a tangent, is perpendicular to the tangent.*

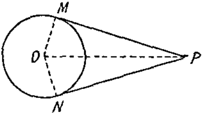


FIG. 38

This follows from (1) above if the point of contact of the tangent is looked upon as an infinitely short chord of the circle.

From this theorem it can be shown that *the two tangents drawn to a circle from any external point are equal in length.*

Thus, referring to Fig. 38, PM and PN are two tangents drawn from the point P to the circle. If the lines MO, NO and PO are drawn the two triangles OMP and ONP are equal in all respects, since

$OM = ON$ . . . . (radii)

OP is common

and $O\hat{M}P = O\hat{N}P = 90°$.

$\therefore$ The tangents PM and PN are equal.

EXAMPLE. A cylinder of diameter 4 ft. is suspended from two points in a horizontal beam by two vertical rope slings, its axis being horizontal and lying 4 ft. below the beam. Calculate the length of each rope.

The arrangement is illustrated in Fig. 39. Now in Fig. 39 (*b*), in which M and N are the points at which the rope breaks contact with the cylindrical surface, the length $OP =$ 4 ft., length $OM = 2$ ft., and, from the foregoing theorem, $O\hat{M}P = 90°$, since PM is obviously a tangent to the circle whose centre is O.

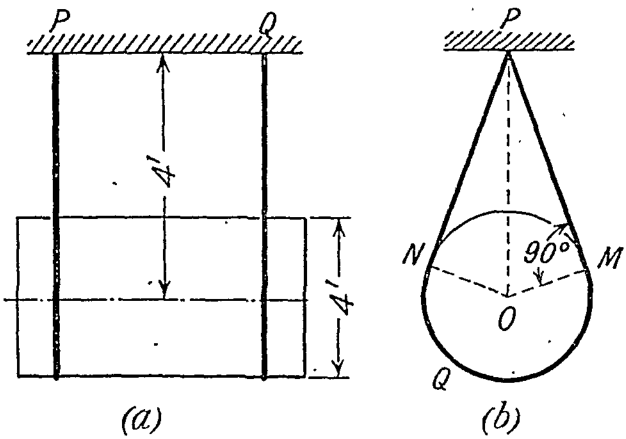


FIG. 39

Referring to the triangle OMP we have, from Pythagoras's Theorem,

$$OP^2 = OM^2 + MP^2$$
$$4^2 = 2^2 + MP^2$$
or
$$MP^2 = 4^2 - 2^2 = 12$$
$$MP = \sqrt{12} = 3{\cdot}46$$
$$= NP.$$

Before we can find the total length of rope we must find the length of the arc NQM, and to do this we must first calculate the angles PON and POM, which will obviously be equal.

$$\cos P\hat{O}M = \frac{OM}{OP} = \frac{2}{4} = 0{\cdot}5$$
$$\therefore \quad P\hat{O}M = 60°$$
$$= P\hat{O}N$$
$$\therefore \quad P\hat{O}M + P\hat{O}N = 120°$$

$$\therefore \qquad \frac{\text{Arc } NQM}{\text{Circumference of circle}} = \frac{240°}{360°} = \frac{2}{3}.$$

Hence $\qquad$ Arc $NQM = \frac{2}{3} \times \pi \times 4$

$$= \frac{8}{3}\pi = 8{\cdot}38 \text{ ft.}$$

Thus total length of rope

$$= 8{\cdot}38 + 2 \times 3{\cdot}46$$
$$= 15{\cdot}3 \text{ ft.}$$

4. *If two circles intersect, the line of centres bisects the common chord at right angles.*

Two intersecting circles are shown in Fig. 40, their centres

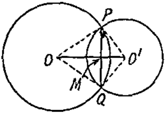


FIG. 40

being O and O'. Since they intersect at P and Q the line PQ is the common chord. OO' is, of course, the line of centres.

Both O and O' are joined to P and Q. Then in the triangles OPO' and OQO' so formed we have

$OP = OQ$ . . . . (radii)

$O'P = O'Q$ . . . . (radii)

and OO' is common.

Hence these triangles are equal in all respects, and therefore $P\hat{O'}O = Q\hat{O'}O$. Taking, now, triangles $PO'M$ and $QO'M$ we have

$O'P = O'Q$ . . . (radii)

$P\hat{O'}M = Q\hat{O'}M$ . . . (proved above)

$O'M$ common.

Thus these triangles are also equal in all respects, and hence

$PM = MQ$

and $\quad P\hat{M}O' = Q\hat{M}O'$,

which means that the common chord PQ is bisected at right angles by the line of centres OO'.

It follows from the above, by considering the point of contact as an infinitely short common chord, that *if two circles touch each other, the line of centres passes through the point of contact*; either directly, as in Fig. 41 (a), or when produced

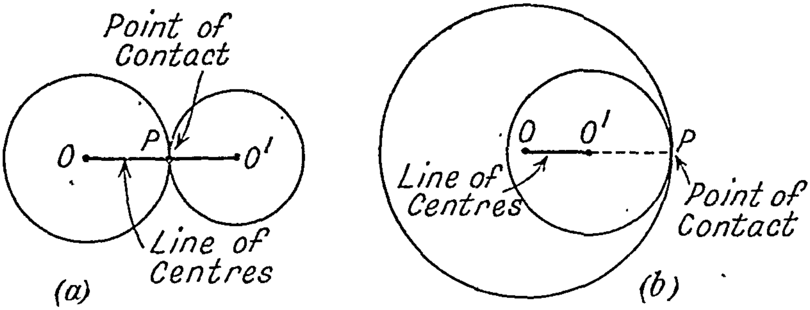


FIG. 41

as in Fig. 41 (b). In the second case one of the circles lies inside the other.

EXAMPLES XXII

(1) A cylindrical tank whose axis is horizontal has a diameter of 6 ft. and an axial length of 8 ft. 6 in. Water is run into it until the water surface is 1 ft. from the top. Calculate the area of the surface of the water.

(2) A sphere of diameter 3 in. is cut through, the cutting plane being $\frac{1}{2}$ in. from the centre of the sphere. Calculate the area of the cut surface.

(3) A cylindrical boiler is laid with its axis horizontal along the centre of a platform 10 ft. wide. The boiler is 6 ft. in diameter and is fastened down by chains passing over it, the ends of the chains being attached to the outer edges of the platform. Calculate the length of one of these chains.

(4) A cylinder of diameter 3 ft. is laid with its axis horizontal on two horizontal bars, parallel to each other and to the axis of the cylinder, whose centres are 2 ft. apart and 1 ft. above the ground. Calculate the height of the top of the cylinder above ground if the diameter of each bar is 4 in.

(5) Calculate the length of the common chord of two circles, radii 3 in. and 2 in. respectively, when their centres are 4 in. apart.

(6) Three equal cylinders of radii 1 ft. are bound tightly together by a rope which passes once round them at right angles to their axes. Calculate the length of the rope.

(7) A circle of radius 3 cm. touches two straight lines which are inclined at an angle of 50°. Calculate the area enclosed between the circle and the lines.

(8) Three equal circles of radii 2 in. touch each other. Calculate the area enclosed between them.

(9) Prove that six circles equal in radius to a given circle can be arranged so that each touches the given circle and two others of the six.

(10) Four equal circles are arranged so that each touches two others of the four and also a circle of radius 1 in. Calculate their radii.

5. *The angle subtended at the centre of a circle by an arc is double the angle subtended at the circumference by the same arc.*

Referring to Fig. 42, the angle POQ is subtended at the centre of the circle by the arc PMQ, and the angle PXQ is the angle subtended at the circumference of the circle by the same arc, X being any point on the major arc PXQ.

We shall show that $P\hat{O}Q = 2\,P\hat{X}Q$. Before we can prove this, however, we must consider a property of an *exterior* angle of a triangle. In Fig. 42 (*b*) the side CB of the triangle ABC is produced to D. Then it can be shown that

$$D\hat{B}A = B\hat{A}C + B\hat{C}A,$$

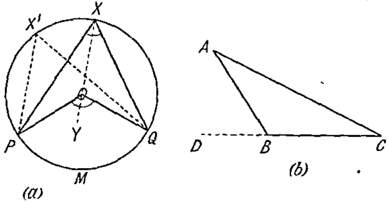


Fig. 42

or, in words, the exterior angle is equal to the sum of the interior opposite angles.

Now, in Fig. 42 (*a*), if we draw the line XO and produce it to Y we can use this property of the exterior angle and say that

$$P\hat{O}Y = O\hat{P}X + O\hat{X}P \text{ in the triangle } POX$$

and, also, $Q\hat{O}Y = O\hat{Q}X + O\hat{X}Q$ in the triangle QOX.

$$\therefore P\hat{O}Q = P\hat{O}Y + Q\hat{O}Y = O\hat{P}X + O\hat{X}P + O\hat{Q}X + O\hat{X}Q.$$

But both of the triangles POX and QOX are isosceles since two of their sides are radii, so that their base angles are equal. Thus

$$O\hat{P}X = O\hat{X}P$$

and $$O\hat{Q}X = O\hat{X}Q,$$

so that we can write

$$P\hat{O}Q = 2\,O\hat{X}P + 2\,O\hat{X}Q = 2\,(O\hat{X}P + O\hat{X}Q) = 2\,P\hat{X}Q.$$

Now it was stated that X was *any* point on the major arc, so that if instead we had taken the point X' we could have proved in exactly the same way that

$$P\hat{O}Q = 2\,P\hat{X}'Q.$$

Hence, angles PXQ and $PX'Q$ must be equal. This means that *angles subtended at the circumference by the same arc* (*i.e. angles in the same segment*) *are equal.* †

Further, in the special case when the arc PMQ becomes a semicircle as in Fig. 43—P and Q being then the ends of a diameter POQ—the angle POQ is now 180°, and $P\hat{X}Q$ is always

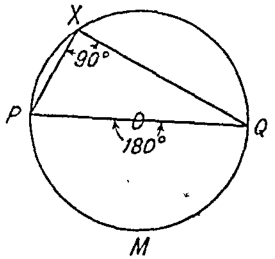


FIG. 43

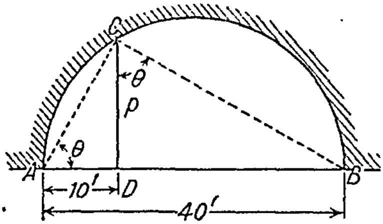


FIG. 44

half of this (as shown above), so that $P\hat{X}Q = 90°$, and all other angles drawn in the upper semicircle have the same value.

Otherwise stated, "*the angle in a semicircle is a right angle.*"

EXAMPLE. A bridge over a canal is in the form of a semicircular arch of 40 ft. diameter. A barge passes under the bridge at a distance of 10 ft. from one of the end supports of the arch. Calculate the greatest height above this end support which the mast of the barge can have if it is to clear the arch.

In Fig. 44 the vertical line DC, of height p ft., represents the mast of the barge. The point C is joined to the two ends A and B of the arch.

Let $\qquad C\hat{A}D = \theta.$

Since CD is vertical $C\hat{D}A = 90°.$

$\therefore \qquad A\hat{C}D = 90 - \theta.$

† Our proof is given for the simplest case only. The result holds for all cases, including the one when PXQ is a minor arc.

But we have proved above that the angle in a semicircle is 90°, so that $A\hat{C}B = 90°$,

$$\therefore \quad D\hat{C}B = A\hat{C}B - A\hat{C}D$$
$$= 90° - (90 - \theta)° = \theta.$$

Now, in triangle CDA,

$$\frac{p}{10} = \tan\theta$$

and in triangle CDB

$$\frac{BD}{p} = \frac{30}{p} = \tan\theta$$

$$\therefore \quad \frac{p}{10} = \frac{30}{p} \text{ since both are equal to } \tan\theta$$

or
$$p^2 = 300$$
$$p = \sqrt{300} = 17{\cdot}3 \text{ ft.}$$

(The student might usefully check this graphically.)

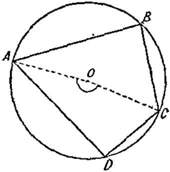


FIG. 45

From the fact that the angle subtended by an arc of a circle at the centre is twice the angle at the circumference it follows that—

The sum of each pair of opposite angles in a cyclic quadrilateral (i.e. a four-sided figure drawn within a circle with all its four corners lying on the circle) is 180°.

Referring to Fig. 45 $ABCD$ is a cyclic quadrilateral. If we join the centre O of the circle to A and C we have the marked angle AOC equal to 2 $A\hat{B}C$ and the unmarked angle AOC

equal to $2\,ADC$. But the marked and unmarked angles AOC are together equal to 360°, so that we can say

$$360° = 2A\hat{B}C + 2A\hat{D}C$$

or

$$180° = A\hat{B}C + A\hat{D}C,$$

and since $A\hat{B}C$ and $A\hat{D}C$ are opposite angles of the quadrilateral the above statement is proved for one pair of opposite angles. If O were joined to B and D it could be shown in exactly the same way that the sum of the other pair of opposite angles

$$D\hat{A}B + D\hat{C}B = 180°.$$

Thus, for example, if $D\hat{A}B = 50°$ then $D\hat{C}B = 180° - 50° = 130°$.

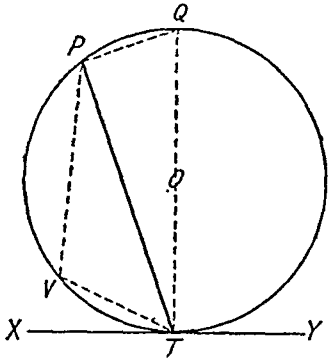


FIG. 46

6. *The angles between a tangent to a circle and a chord drawn from the point of contact of the tangent are equal to the angles in the alternate segments of the circle.*

Fig. 46 illustrates this theorem. XY is a tangent to the circle touching it at T. TP is a chord. From the above statement $X\hat{T}P$ is equal to any angle drawn in the segment PQT (all such angles being equal as proved in a previous paragraph), and also $Y\hat{T}P$ is equal to any angle drawn in the segment PVT.

In the figure the line TQ is a diameter, and it is sufficient to prove that $X\hat{T}P = P\hat{Q}T$, since $P\hat{Q}T$ equals any other angle in this segment.

Now, since QT is a diameter, $X\hat{T}Q = 90°$ (since it is the angle between a tangent and a radius) and, also, $Q\hat{P}T = 90°$ (since it is the angle in a semicircle)

$$\therefore \qquad X\hat{T}P + P\hat{T}Q = 90°$$

$$\text{and} \qquad P\hat{Q}T + P\hat{T}Q = 90°,$$

which means that $X\hat{T}P = P\hat{Q}T$, since the angle $P\hat{T}Q$ occurs in both equations. Again, $Y\hat{T}P = 180° - X\hat{T}P$, but from

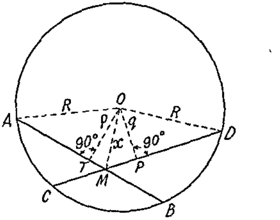


Fig. 47

the properties of a cyclic quadrilateral we can say, regarding the cyclic quadrilateral $T\hat{V}PQ$, that

$$T\hat{V}P + P\hat{Q}T = 180°$$

$$\text{or} \qquad T\hat{V}P = 180° - P\hat{Q}T.$$

$$\text{But} \qquad P\hat{Q}T = X\hat{T}P$$

$$\therefore \qquad T\hat{V}P = 180° - X\hat{T}P$$

$$= Y\hat{T}P.$$

7. *If two chords of a circle intersect within the circle the product of the two sections of one of them is equal to the product of the two sections of the other.*

Referring to Fig. 47, AB and CD are two intersecting chords of a circle of radius R and centre O.

Then $$AM \times MB = DM \times MC.$$

Proof. Join the centre O to A, M and D, and draw perpendiculars OT and OP from O on to the two chords. Let the lengths of these perpendiculars be p and q, and let the length $OM = x$.

Now $$AM = AT + TM$$
and $$MB = TB - TM.$$
Since the perpendicular OT bisects the chord AB, $AT = TB$, and we may write
$$MB = AT - TM.$$
Similarly $$DM = DP + PM$$
and $$MC = CP - PM.$$
But $$CP = DP,$$
$\therefore$ $$MC = DP - PM.$$

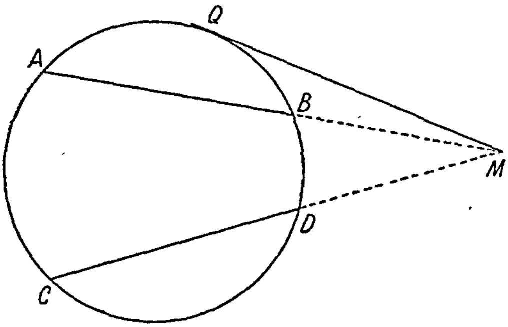


FIG. 48

Hence
$$\begin{aligned} AM \,.\, MB &= (AT + TM)(AT - TM) \\ &= AT^2 - TM^2, \end{aligned}$$
and
$$\begin{aligned} DM \,.\, MC &= (DP + PM)(DP - PM) \\ &= DP^2 - PM^2. \end{aligned}$$

Now, using Pythagoras's Theorem, in triangle ATO, $AT^2 = R^2 - p^2$, and in triangle MTO, $TM^2 = x^2 - p^2$,

$\therefore$ $$\begin{aligned} AT^2 - TM^2 &= R^2 - p^2 - (x^2 - p^2) \\ &= R^2 - x^2. \end{aligned}$$
Similarly $$DP^2 = R^2 - q^2$$
and $$PM^2 = x^2 - q^2$$
$\therefore$ $$\begin{aligned} DP^2 - PM^2 &= R^2 - q^2 - (x^2 - q^2) \\ &= R^2 - x^2 \\ &= AT^2 - TM^2. \end{aligned}$$
But $$AT^2 - TM^2 = AM \,.\, MB$$
and $$DP^2 - PM^2 = DM \,.\, MC$$
$\therefore$ $$AM \,.\, MB = DM \,.\, MC.$$

It can also be proved that if the two chords do not intersect internally but do so externally if they are produced, there is

still equality between the products of the sections. Thus in Fig. 48 AB and CD are two chords which, when produced, intersect at M external to the circle.

It can be shown that, in this case also,

$$AM \cdot MB = DM \cdot MC.$$

Further, each of these products can be proved equal to the square of the tangent, MQ, from M to the circle, so that we may write

$$AM \cdot MB = DM \cdot MC = MQ^2.$$

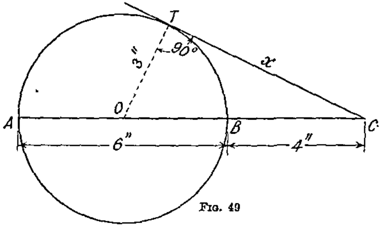


FIG. 49

EXAMPLE. A diameter AB of a circle of radius 3 in. is produced to a point C such that $AC = 10$ in. Calculate the length of the tangent drawn from C to the circle.

Length $AB = 6$ in. since it is a diameter. Length $AC = 10$ in. Therefore length $BC = 4$ in.

Let x = length of the tangent from C to the circle.

Then $\qquad AC \times CB = x^2$

or $\qquad 10 \times 4 = x^2$

$$40 = x^2$$

$$6{\cdot}32 = x.$$

$\therefore$ Length of the tangent is 6·32 in.

The problem is illustrated in Fig. 49, and the above working can be checked in another way. The point of contact T of the tangent from C is joined to the centre O. Then $OT = 3$ in.

and OC is obviously 7 in. Again, the radius OT is at right angles to the tangent CT. Then from Pythagoras's Theorem,

$$OC^2 = OT^2 + TC^2$$
$$7^2 = 3^2 + TC^2$$
$$49 = 9 + TC^2$$
$$40 = TC^2$$
$$TC = \sqrt{40} = 6{\cdot}32 \text{ in.}$$

which is the same result as before.

EXAMPLES XXIII

(1) Calculate the angle subtended at the circumference of a circle of radius 2 in. by an arc of length 1·5 in.

(2) In a circle of radius 5 cm. a triangle is drawn, its three corners lying on the circle. One of the sides is 10 cm. long and another is 6 cm. Calculate the length of the third side.

(3) A quadrilateral $ABCD$ is drawn in a circle of centre O, the four corners lying on the circle. $D\hat{A}B = 60°$ and $A\hat{B}C = 80°$. If the points A, B, C and D are joined to the centre O, calculate the larger angle DOB and the larger angle AOC.

(4) A tangent is drawn to a circle at a point A and two chords AC and AB are drawn, these making angles of 60° and 110° respectively with the tangent. If the points B and C are joined calculate the angle BCA.

(5) Two tangents PM and PN are drawn to a circle from an external point P. Prove that if M and N are joined, the chord MN is bisected by the line joining P to the centre of the circle.

(6) An open belt passes round two pulleys whose centres are 8 ft. apart. The distance between the point at which the belt leaves one pulley and the point at which it meets the other is 7 ft. 6 in. If the radius of the smaller pulley is 1 ft. calculate the radius of the larger pulley and the total length of the belt.

(7) The centres of two pulleys, of radii 7 in. and 2 in., are 13 in. apart. An open belt passes round the pulleys. Calculate its length.

(8) A crossed belt passes round two pulleys whose centres are 13 in. apart. The distance between the point at which the belt leaves one pulley and the point at which it meets the other is 12 in. If the radius of the smaller pulley is 2 in., calculate the radius of the larger pulley and the total length of the belt.

(9) The centres of two pulleys of radii 9 in. and 4 in. are 17 in. apart. A crossed belt passes round the pulleys. Calculate its length.

(10) A circular disc of radius 2 ft. is suspended by two strings AB and AC where BC is a chord of the disc 10 in. long. If the lengths of the strings are both 15 in., calculate the depth of the centre of the disc below the point A.

(11) The radius of a circle is 12 in. and the length of a chord AB is 8 in. If the tangents to the circle at A and B meet at T, calculate the length of TA.

(12) Taking a circular disc of radius 2 ft. suspended by equal strings attached at the ends of a chord of length 10 in., as in Question 10, calculate the shortest lengths of the strings if they are to be straight.

Equiangular and Similar Figures. We have now to consider the very important properties of *equiangular* and *similar* triangles and polygons. Space will not permit us to prove the

statements made, but the various properties of such figures should be thoroughly understood and remembered.

Two polygons are equiangular if they have the same number of sides and if the angles of one taken in order are equal respectively to the angles of the other taken in order.

If, in addition, all pairs of corresponding sides (i.e. sides adjacent to equal angles) of two equiangular polygons have the same ratio to one another, the polygons are said to be similar.

It should be noted that there is an important difference between similar polygons and polygons which are only equiangular. Thus, in Fig. 50 (*a*) the two quadrilaterals are similar

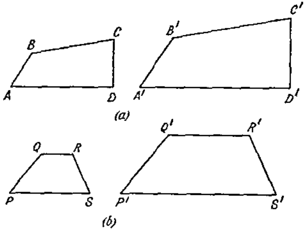


FIG. 50

because, in addition to being equiangular ($B\hat{A}D = B'\hat{A}'D'$, $A\hat{B}C = A'\hat{B}'C'$, etc.), if we take the ratios of corresponding sides we have

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA} = \frac{3}{2}.$$

In Fig. 50 (*b*) the two quadrilaterals are equiangular but they are not similar, since

$\frac{P'Q'}{PQ}$ is *not* the same as the ratio $\frac{Q'R'}{QR}$ (for example), i.e. the ratios of corresponding sides are not all the same.

In the *case of triangles*, however, if the *triangles are equiangular they are also similar.*

Thus, in Fig. 51, ABC and $A'B'C'$ are equiangular (i.e.

$C\hat{A}B = C'\hat{A}'B'$; $A\hat{C}B = A'\hat{C}'B'$; $C\hat{B}A = C'\hat{B}'A'$) and it therefore follows, from the above statement, that they are similar —i.e.

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}.$$

This property of equiangular triangles is very important in some practical problems.

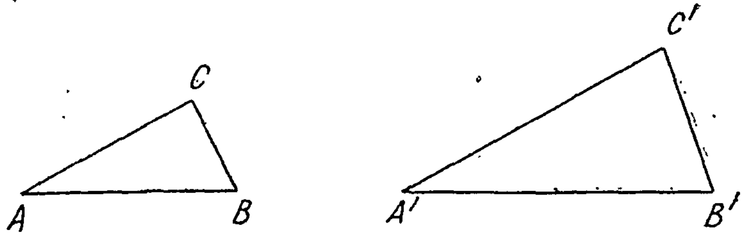


FIG. 51

EXAMPLE. A useful application of the properties of similar triangles is the method of dividing a straight line into a given number of equal parts by a graphical construction.

Referring to Fig. 52, suppose we wish to divide the line AB into three equal parts. We draw a line such as AX making any

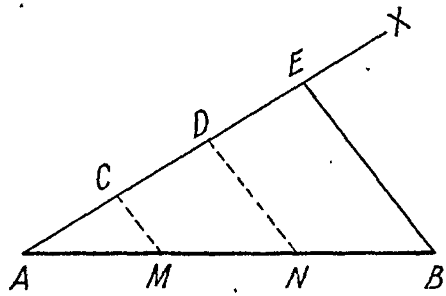


FIG. 52

angle with AB, and mark off along it any three suitable *equal* lengths AC, CD and DE. Join E to B and draw DN and CM parallel to EB to meet AB in N and M. Thus AM, MN and NB are the three equal parts of the line AB.

This follows from the fact that the three triangles ACM, ADN and AEB are equiangular, and therefore similar.

Hence their corresponding sides are proportional, so that

$$\frac{AC}{AD} = \frac{AM}{AN} = \frac{1}{2}$$

and $$\frac{AC}{AE} = \frac{AM}{AB} = \frac{1}{3}.$$

Thus $$AM : AN : AB = 1 : 2 : 3.$$

(The sign : denotes proportionality.)

It can also be shown *for both triangles and polygons that if two figures are similar their areas are in the same proportion as the squares of corresponding sides.*

EXAMPLE. An electric transmission line joins five towns and returns to the town from which it started. The towns and the line are shown on a map which is drawn to the scale of $\frac{1}{2}$ in. to the mile. If the area enclosed within the loop formed by the line on the map is 53 sq. in. what is the actual area enclosed in square miles?

Let the length of one side of the figure on the map be x in. This represents $2x$ miles.

Then
$$\frac{\text{Actual area in sq. miles}}{\text{Map area in sq. in.}} = \frac{(2x)^2}{x^2} = \frac{4x^2}{x^2} = \frac{4}{1};$$

$$\therefore \quad \text{Actual area in square miles} = 4 \times 53$$
$$= 212.$$

(Note that the above working does not involve the number of sides which the figure has. The two similar areas are proportional to the squares of corresponding sides, no matter how many sides there may be.)

Similar Solids.

The volumes of similar solids are in the same proportion as the cubes of their corresponding linear dimensions.

EXAMPLES XXIV

(1) Prove that the lengths of the perpendiculars drawn to the bases of two similar triangles from the opposite vertices are proportional to corresponding sides of the two triangles.

(*Note.* A *vertex* (*pl. vertices*) is an angular point of a triangle.)

(2) A triangle of area A sq. in. is divided into two parts by a line drawn parallel to one side, the distance of this line from the side being half its distance from the opposite vertex of the triangle. Show that the triangular part is similar to the original triangle and hence calculate the areas of the two parts in terms of the area A. (Use the information given in Example I for the calculation.)

(3) Two poles, one twice as long as the other, are set up in a vertical plane

with their upper ends fastened together, and their lower ends are 40 ft. apart in a horizontal plane. A chain hangs down from the joining point of the poles and extends to the ground. Calculate the length of the chain and the distance of its lower end from each of the lower ends of the poles if the angle between the poles is 90°.

(4) The county of Cheshire has an area of 1004 square miles. What area, in square inches, will it occupy on a map drawn to the scale of 10 miles to the inch?

(5) A lamp is placed at a distance of 10 in. from a metal sheet in which a circular hole, of diameter 2 in., is cut. What will be the area of a circle of light thrown on a screen which is placed on the opposite side of the metal sheet and parallel to it and at a distance of 4 ft. from the lamp?

(6) A man can just see the top of a mountain over the roof of a house which stands between him and the mountain. If he knows that the mountain is 430 times as high as the house, how far is he from the mountain if his distance from the house is 120 ft.?

(7) A man stands with his stick held beside him. The height of the man is 5 ft. 10 in. and the length of the stick is 2 ft. 11 in. If the man's shadow is 8 ft. 2 in., how long is the shadow of the stick?

(8) The radii of two circles whose centres are O and O' are in the proportion 3 : 2. From a point on the line through $O\ O'$ a common tangent is drawn to the two circles. If the centres are 6 in. apart calculate the distance from O of the point from which the common tangent is drawn.

(9) A model of a yacht is on the scale of $\frac{1}{2}$ in. to the foot. If the area of a sail of the model is 180 sq. in., what in sq. yd. is the area of the corresponding sail of the yacht?

(10) The model of a bridge is 16 in. long and its total surface area is 73 sq. in. If the actual bridge is 200 ft. long, what in sq. ft. is its total surface area?

(11) An exact scale model of a ship is $\frac{1}{50}$th full size. What is the ratio of the weight of the model to that of the ship if the same materials are used for both?

(12) A triangle ABC is right-angled at A and $AB = 5$ in., $AC = 12$ in.; D is the foot of the perpendicular from A on to BC. Prove that the triangles ABC, DBA, DCA are equiangular and calculate the perpendicular distances of D from AB and AC.

Surfaces and Volumes of Complex Solid Bodies

We have already dealt with the calculation of the surface and volume of various simple solid bodies in Book I. We must now consider bodies of a rather more complex nature.

Frustum of a Cone or Pyramid. If the upper portion of a cone or pyramid is removed by cutting it by a plane parallel to the base, the remaining lower portion is called a *frustum* of the cone or pyramid. This is illustrated in Fig. 53, in which a cone, whose base radius is R and whose perpendicular height is H, is divided into two parts by being cut through in a plane parallel to its base and at a height h above the base. Thus the frustum is of perpendicular height h and slant height s, and the radius of its upper surface is r.

Now the triangles ABM and ACN are obviously equiangular and, therefore, similar.

Hence $$\frac{AB}{AC} = \frac{r}{R}$$

or $$\frac{H-h}{H} = \frac{r}{R}.$$

Thus $$r = \frac{H-h}{H} . R.$$

Volume of Frustum. The volume of the frustum can be found by treating it as the difference in volume between that of the original cone, of height H, and that of the cone of height $H-h$, which has been removed.

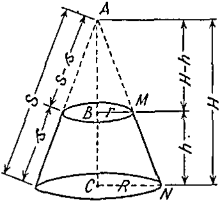


FIG. 53

Thus $$V = \tfrac{1}{3}\pi R^2 H - \tfrac{1}{3}\pi r^2(H-h).$$

Now $$r = \frac{H-h}{H} . R = \left(1 - \frac{h}{H}\right) . R = R - \frac{h}{H}R$$

from which $H = \dfrac{Rh}{R-r}$.

If we substitute for H in the original formula we have

$$\text{Volume of frustum} = \tfrac{1}{3}\pi\left[R^2 . \frac{Rh}{R-r} - r^2\left(\frac{Rh}{R-r} - h\right)\right]$$

which, after simplification, gives

$$\text{Volume of frustum} = \tfrac{1}{3}\pi h\,(R^2 + Rr + r^2).^*$$

* This can also be written $\dfrac{h}{3}(A + \sqrt{Aa} + a)$, where A and a are the extreme cross-sectional areas. This formula also applies to a frustum of a cone in which the cross section is not circular.

In working examples, however, it is perhaps better to work from first principles rather than to rely on this formula, since the formula can easily be forgotten, while the method of calculation based on the difference in volume of two cones is quite straightforward.

Curved Surface of Frustum. In the same way, the curved surface of the frustum is the difference in curved surface of two cones, one of height H and slant height S, and the other of height $H-h$ and slant height $S-s$.

Thus $$C = \pi RS - \pi r\,(S-s).$$

Again, from similar triangles,

$$\frac{S-s}{S} = \frac{H-h}{H} = \frac{r}{R},$$

from which the curved surface can be shown to be

$$= \pi\, s\,(R+r).$$

The volume and surface of the frustum of a pyramid can be found by methods similar to the above.

EXAMPLE. The perpendicular height of the frustum of a pyramid is 4 in., its base and top being squares whose sides are 6 in. and 2 in. respectively. Calculate the volume and slant surface of the frustum.

Treating the frustum as the lower portion of a pyramid of perpendicular height H we have, from similar triangles (see Fig. 54),

$$\frac{H-4}{1} = \frac{H}{3}$$

from which

$$3H - 12 = H$$

or

$$H = 6$$

$$\therefore \quad \text{Volume of frustum} = \tfrac{1}{3}\,.\,6^2\,.\,H - \tfrac{1}{3}\,.\,2^2\,.\,(H-4)$$
$$= \tfrac{1}{3}\,.\,6^2\,.\,6 - \tfrac{1}{3}\,.\,2^2.\,2$$
$$= 69\tfrac{1}{3} \text{ cub. in.}$$

Again, from Pythagoras's Theorem,

$$OA^2 = OM^2 + MA^2$$
$$= 6^2 + 3^2 = 45$$
$$OA = \sqrt{45} = 6{\cdot}7 \text{ in.}$$

Now, from similar triangles,

$$\frac{OB}{OA} = \frac{1}{3}$$

$$\therefore \qquad OB = \tfrac{1}{3} \times 6{\cdot}7 = 2{\cdot}23 \text{ in.}$$

Hence the area of each slant side of the frustum

$$= \tfrac{1}{2}\,.\,6\,.\,OA - \tfrac{1}{2}\,.\,2\,.\,OB$$
$$= 20{\cdot}1 - 2{\cdot}2$$
$$= 17{\cdot}9 \text{ sq. in.}$$

$$\therefore \quad \text{Total slant surface} = 4 \times 17{\cdot}9$$
$$= 71{\cdot}6 \text{ sq. in.}$$

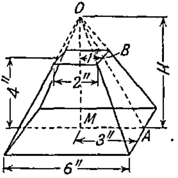


FIG. 54

Zone of a Sphere. If two parallel planes cut through a sphere the portion of the sphere lying between them is called a *zone* of the sphere.

Two such zones are shown in Fig. 55, the upper one being referred to as a *spherical cap* because the second intersecting plane in this case is tangential to the sphere.

It can be shown that the curved surface of a zone of a sphere is $2\pi Rh$

where R = radius of the sphere

h = height of the zone

= distance between the two planes intersecting the sphere.

In the same way, the curved surface of the spherical cap is $2\pi RH$, where H is its height.

A special case of the spherical cap is the *hemisphere*, when

$H = R$. From the above formula, therefore, the surface of a hemisphere is $2\pi R \,.\, R = 2\pi R^2$, which agrees with the formula for the surface of a sphere—

$$S = 4\pi R^2$$

given in Book I.

The volume of a spherical cap can be obtained from the formula for a sector of a sphere as shown in Fig. 56.

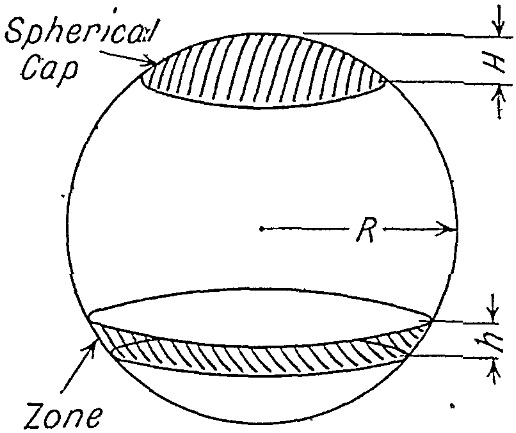


FIG. 55

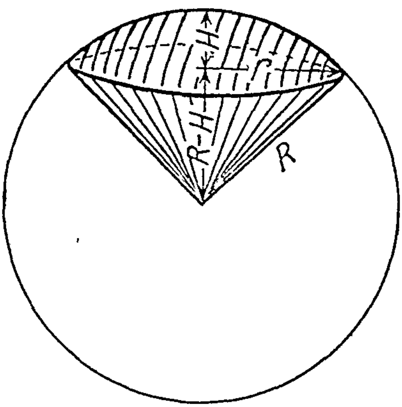


FIG. 56

Now it can be shown that the volume of a sector of a sphere

$$= \tfrac{2}{3}\,\pi R^2 H,$$

where H is the height of the spherical cap as before.

The sector obviously consists of a spherical cap, together

with a cone of perpendicular height $R-H$, whose base radius is r. The volume of this cone is $\frac{1}{3}\pi \cdot r^2 \cdot (R-H)$ so that the volume of the spherical cap alone is

$$\tfrac{2}{3}\pi R^2 H - \tfrac{1}{3}\pi r^2 (R-H).$$

From Pythagoras's Theorem

$$R^2 = (R-H)^2 + r^2.$$

By substituting for r^2 from this equation, we obtain for the volume of a spherical cap

$$\text{Volume of cap} = \pi H^2 \left(R - \frac{H}{3}\right).$$

The volume of a zone which is *not* a spherical cap may be found by treating its volume as the difference in volume between two spherical caps whose difference in height is the height of the zone in question.

EXAMPLE. Calculate the curved surface and volume of the zone of a sphere of radius 5 in. which lies between two parallel planes whose distances from the centre of the sphere are 1 in. and 2 in. respectively.

The height of the zone is 1 in.

$$\begin{aligned} \therefore \text{ Surface of zone } &= 2\pi Rh \\ &= 2\pi \cdot 5 \cdot 1 \\ &= 10\pi \\ &= 31{\cdot}42 \text{ sq. in.} \end{aligned}$$

Volume of zone

= Volume of a spherical cap of height 4 in. − Volume of a spherical cap of height 3 in.

$$\begin{aligned} &= \pi \cdot 4^2 (5 - \tfrac{4}{3}) - \pi \cdot 3^2 (5 - \tfrac{3}{3}) \\ &= \pi \cdot 16 \cdot 3\tfrac{2}{3} - \pi \cdot 9 \cdot 4 \\ &= 22\tfrac{2}{3}\pi \\ &= 71{\cdot}22 \text{ cub. in.} \end{aligned}$$

EXAMPLES XXV

(1) A metal bucket is 12 in. deep, the bottom being 8 in. diameter and the top 12 in. What weight of water will the bucket hold if 1 cub. ft. of water weighs 62·3 lb.? Calculate also the area of metal sheet which is required to make the bucket.

(2) A cone of perpendicular height 12 in. and whose base diameter is 8 in. is divided into two parts by a cut parallel to the base and at a height of 7 in. above the base. Calculate the volumes of the two parts.

(3) The radii of the ends of a frustum of a cone are 1 in. and $\frac{1}{2}$ in. and they are 3 in. apart. The smaller ends of two such frustums are fastened together to form a bobbin. Calculate its total surface area and its volume.

(4) A tapering interplane strut of an aeroplane has a central portion 34 in. long with uniform cross-sectional area of 4·2 sq. in. The portion at each end is a frustum of a cone with extreme cross-sectional areas of 4·2 sq. in. and 2·1 sq. in. which are 17 in. apart. Calculate the volume of the strut.

(5) Two cones have the same vertex and axis and common height 5 cm. The base diameters are 4 cm. and 3 cm. The lower part is cut off by a plane parallel to the base and 2 cm. from it to form a conical washer. Find its volume.

(6) A stone column is 20 ft. high. The diameter of its circular base is 3 ft. and the diameter of its top is 2 ft. Calculate the weight of the column if the stone of which it is made weighs 148 lb. per cub. ft.

(7) The floor of an ornamental pool is in the form of a cap of a sphere whose radius is 30 ft. The depth of the pool at the centre is 3 ft. Calculate the number of cubic feet of water which it will hold and also the diameter of the pool at its surface.

(8) A gas-holder is in the form of a cylinder with a spherical cap as its top. The height of the cylindrical part is 40 ft. and its diameter is 50 ft. The top is the cap of a sphere whose radius is 60 ft. Calculate the height of the centre of the top above the base of the holder, and calculate also the number of cubic feet of gas which the gas-holder will contain.

(9) The head of a carriage bolt consists of a spherical cap. The radius of the plane face is 1·2 in. and the height of the cap is 0·3 in. Calculate the area of the spherical surface and the volume.

(10) A cylindrical hole of radius 5 cm. is drilled through a sphere of radius 13 cm. The axis of the hole passes through the centre of the sphere. Calculate the total surface area and the volume of the part remaining.

(11) A dumb-bell is made up of a cylinder 4 in. long and $1\frac{1}{2}$ in. diameter and end pieces consisting of the major portions of spheres of diameters $2\frac{1}{2}$ in. and cut by planes to make sections equal to that of the cylinder. Find the overall length, the surface area and the volume of the dumb-bell.

Surface, Volume, and Space Factor of Former-wound Coils. In electrical engineering work it is often necessary to be able to calculate the surface area, the volume, etc., of coils of wire which have been wound on a bobbin or *former*. A few examples of such calculations will now be given.

EXAMPLE 1. Calculate the total surface area and volume of a coil of length 15 cm. wound on a cylindrical former of diameter 20 cm., the depth of the winding being 4 cm.

The coil is shown in Fig. 57 (*a*).

The outer cylindrical surface

$$= 2\pi RH \text{ (see Chapter VI, Book I).}$$

$$= \pi \times 28 \times 15 = 1320 \text{ sq. cm.}$$

Inner cylindrical surface

$$= \pi \times 20 \times 15 = 943 \text{ sq. cm,}$$

Surface of each end

= the difference in area between two circles of diameters 28 and 20 cm.

$$= \frac{\pi \times 28^2}{4} - \frac{\pi \times 20^2}{4} = 616 - 314$$

$$= 302 \text{ sq. cm.}$$

$\therefore$ Total end surface = 604 sq. cm.

Hence total surface of the coil

$$= 1320 + 943 + 604$$

$$= 2867 \text{ sq. cm.}$$

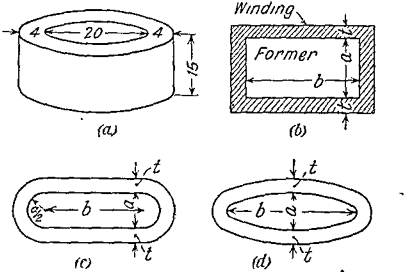


FIG. 57

The volume is the difference in volume between two cylinders of diameters 28 cm. and 20 cm. respectively.

$$\therefore \text{ Volume} = \left(\frac{\pi \times 28^2}{4} \times 15\right) - \left(\frac{\pi \times 20^2}{4} \times 15\right)$$

$$= 15\left(\frac{\pi \times 28^2}{4} - \frac{\pi \times 20^2}{4}\right)$$

$$= 15 \times \text{end surface of coil}$$

$$= 15 \times 302$$

$$= 4530 \text{ cub. cm.}$$

Instead of a cylindrical former, formers whose cross-sections are rectangular, link-shaped, or elliptical, as in Figs. 57 (b), 57 (c), 57 (d), are sometimes used. The calculations of surface and volume may be carried out in a manner almost exactly similar to the above.

If d is the length of the coil (not shown in the figures since these are plan views) and t the depth of winding in each case we have—

Rectangular Former.

Outside surface $= 2\,(a + 2t)\,d + 2(b + 2t)\,d.$

Inner surface $= 2\,ad + 2\,bd.$

Each end surface $= (a + 2t)\,(b + 2t) - a\,.\,b.$

Volume $=$ Surface of one end $\times\, d.$

$= [(a + 2t)\,(b + 2t) - a\,.\,b]\,d.$

Link-shaped Former. In this case the cross-section of the former is a rectangle of width a and breadth b having a semi-circle (radius $a/2$) at each end.

Outside surface $= 2\,b\,.\,d + 2\pi\left(\dfrac{a}{2} + t\right).\,d.$

Inner surface $= 2b\,.\,d + 2\pi\dfrac{a}{2}\,.\,d.$

Each end surface $= b(a + 2t) + \pi\left(\dfrac{a}{2} + t\right)^2 - \left(b\,.\,a + \pi\dfrac{a^2}{4}\right)$

$= t[2b + \pi\,(a + t)]$ when simplified.

Volume $=$ surface of one end $\times\, d$

$= td[2b + \pi\,(a + t)].$

Elliptical Former. The length of the periphery of an ellipse is approximately $2\sqrt{b^2 + 1{\cdot}4674\,a^2}$

where $b =$ length of major axis

$a =$ length of minor axis

(see the inner ellipse of Fig. 57 (d)).

Thus,

Outside surface $= \left\{2\sqrt{(b + 2t)^2 + 1{\cdot}4674\,(a + 2t)^2}\right\}d.$

Inner surface $= \left\{2\sqrt{b^2 + 1{\cdot}4674\,a^2}\right\}d.$

Each end surface is the difference in area between two ellipses. Since the area of an ellipse is $\frac{\pi}{4} \times$ major axis $\times$ minor axis we can write—

$$\text{Each end surface} = \frac{\pi}{4}[(b + 2t)(a + 2t)] - \frac{\pi}{4}ab$$

$$= \frac{\pi}{4}[(b + 2t)(a + 2t) - ab].$$

$$\text{Volume} = d \times \text{area of each end surface}$$

$$= \frac{\pi d}{4}[(b + 2t)(a + 2t) - ab].$$

Space Factor. Fig. 58 (a) shows some of the individual wires of a coil, the total cross-section of the winding space being a rectangle of dimensions a by b. The diameter of the bare wire is d and, when insulated, each wire has an overall diameter of D.

Now with the "square" or "open" piling shown in the figure (each wire standing immediately above the under wires) the number of wires per layer is obviously $\frac{b}{D}$ and the number of layers $\frac{a}{D}$.

Hence, the total number of wires which can be placed in this winding space is $\frac{b}{D} \times \frac{a}{D} = \frac{a \, . \, b}{D^2}$.

The space-factor of the winding is the ratio of the space occupied by the bare wires to the total space.

In this case the cross-section of each wire without insulation is $\frac{\pi d^2}{4}$, so that the total space occupied by metal $= \frac{\pi d^2}{4} \times$ number of wires $= \frac{\pi d^2}{4} \times \frac{a \, . \, b}{D^2}$.

Thus the space-factor is

$$\frac{\frac{\pi d^2}{4} \times \frac{a \, . \, b}{D^2}}{a \, . \, b} = \frac{\pi d^2}{4D^2}.$$

The distance m between the centres of wires in adjacent layers is obviously equal to D in this case. Fig. 58 (c) shows

an arrangement in which an upper wire "beds" between two lower wires. In this case the distance between layers is m_1, and from Fig. 58 (*d*) it can be seen that

$$m_1 = D \sin 60^\circ$$
$$= 0{\cdot}866\, D.$$

Hence, with this "bedded" winding the number of layers

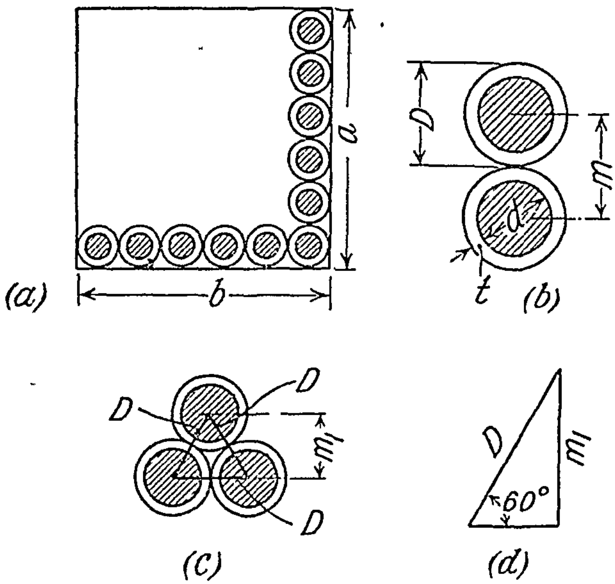


FIG. 58

will be $\dfrac{a}{0{\cdot}866 D}$, instead of $\dfrac{a}{D}$, as in the open winding mentioned above.

Volumes of Bodies whose Cross-sections are Irregular. In Book I we considered the volumes of bodies whose cross-sections were uniform, and of certain other bodies, such as cones and pyramids, for which the cross-section varied along the axis in some uniform and known manner. We now have to consider the determination of the volume of a body, such as a tree-trunk, whose cross-section varies in an irregular way as we move along the axis.

A graphical method has to be used in this case. Perhaps the best way to make the method clear is by an example.

EXAMPLE. Determine the volume of a tree-trunk whose axis is straight and whose cross-section varies along the axis as shown by the following figures—

Distance along the axis (in feet)	0	2	4	6	8	10	12
Cross-section (in square feet)	12·1	11·8	10·9	11·1	10·4	10·0	9·1

A graph is first plotted from these figures as shown in Fig. 59, the points being joined by a smooth curve.

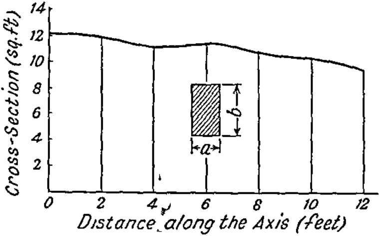


FIG. 59

Now if we consider any area under this graph, such as the shaded rectangle $a \times b$, we see that the area of this rectangle, *to scale*, represents a volume in cubic feet.

Thus if, on the vertical scale of cross-section, 1 in. $= m$ sq. ft., and on the horizontal scale of length along the axis 1 in. represents n ft., then the sides of the rectangle—b in. and a in.—represent bm sq. ft. and an ft. respectively. Hence the area ab, when multiplied by $m \times n$, represents $abmn$ cub. ft.

It follows, therefore, that the total area under the graph, when multiplied by mn, gives the volume of the tree-trunk.

In the figure the area under the graph is 3·7 sq. in., and, also, $m = 8 \cdot 15$ and $n = 4 \cdot 06$ so that the volume of the tree-trunk is

$$3 \cdot 7 \times 8 \cdot 15 \times 4 \cdot 06$$
$$= 122 \cdot 5 \text{ cub. ft.}$$

Determination of the Area under a Graph and its Mean Height—Mid-ordinate Method. In the above example we stated that the area under the graph was 3·7 sq. in. This area could be obtained by counting squares on the graph paper as explained in Book I, page 143.

The *mid-ordinate method* provides a quicker means of carrying out such a determination than that of counting squares.

In Fig. 60 an irregular curve is shown. The area under it is to be determined. The base length L of the curve is divided into a number (n) of equal parts. The larger the number of parts the more accurate will be the determination. At the

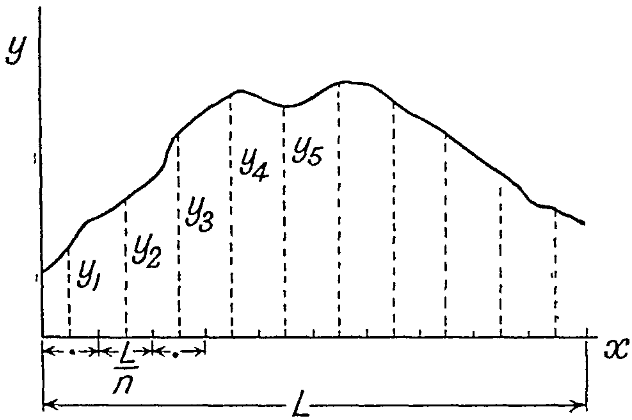


FIG. 60

mid-point of each of these parts an ordinate is set up as shown dotted. These are the *mid-ordinates.*

The mid-ordinates are measured, and their lengths are added together. Their sum is thus

$$y_1 + y_2 + y_3 + \ldots$$

This sum, when divided by the number of parts n into which the base is divided (i.e. the number of mid-ordinates), gives the average, or mean, height of the graph.

Thus

$$\text{Mean height of graph} = \frac{y_1 + y_2 + y_3 + \ldots}{n}$$

The area under the graph is obviously

$$\text{Mean height} \times \text{base} = \left[\frac{y_1 + y_2 + y_3 + \ldots}{n}\right] \times L.$$

Other methods of determining the areas under graphs and the areas of irregular figures will be discussed in Book III.

EXAMPLES XXVI

(1) A field coil is to be wound to fit on a pole of an electric generator. The section of the pole is rectangular and its dimensions are 32 cm. × 28 cm. The length of the coil is 18 cm. and the depth of winding is 4 cm. Calculate its surface area and its volume.

(2) Calculate the surface area and volume of a field coil to fit on a pole of circular cross-section, 25 cm. diameter, the depth of winding being 4 cm. and the length of the coil 20 cm.

(3) A coil is to be wound with circular wire of diameter 0·12 cm., the insulation thickness being 0·015 cm. The winding space is a rectangle 8 cm. wide and 3 cm. deep. If a square, or open, piling is used, calculate the number of turns which can be wound on the coil and, also, the space factor.

(4) Determine the volume of a body whose cross-section varies along its axis as indicated by the following figures—

Distance along axis (in.)	0	1	2	3	4	5	6
Cross-section (sq. in.)	1·7	2·4	2·9	2·6	2·3	3·1	3·4

(Determine the area under the graph both by counting squares and by the mid-ordinate rule.)

(5) The width of a field varies along its length as indicated by the following figures—

Length (yards)	0	20	40	60	80	100	120	140
Width (yards)	55	62	74	69	71	77	82	85

Using the mid-ordinate rule determine the average width of the field and its area.

(6) The speed of a motor-car varies with time as follows—

Time (seconds)	0	2	4	6	8	10	12	14	16
Speed (ft. per sec.)	0	12	22	35	50	65	53	42	33

Using the mid-ordinate method determine the average speed during the 16 sec and also the distance gone in the time.

(*Note.* Area under the graph (to scale) = Feet per second × Seconds = Feet.)

(Additional elementary examples on the work of this chapter will be found on page *178*.)

EXAMPLES XXVII

(1) A beam runs horizontally across a semicircular arch whose diameter is 30 ft. Calculate the length of the beam if it is 11 ft. above the base of the arch.

(2) A pulley of radius 9 in. is fixed with its plane vertical and its centre 2 ft. below a horizontal beam. From a point *A* in this beam, at a distance of 1 ft. 6 in. from the vertical line through the centre of the pulley, a rope passes

round the pulley. Calculate the distance of the point A from the point at which the rope first makes contact with the rim of the pulley.

(3) Two circles whose radii are 5 in. and 3 in. respectively intersect, and the length of the common chord is 4 in. How far apart are their centres?

(4) If the two circles mentioned in Example (3) are actually two very thin discs placed, as stated, on a sheet of paper, calculate the area covered by them when in this position.

(5) A chord AB, of length 3 in., is drawn in a circle whose radius is $2\frac{1}{2}$ in. If the ends of this chord are joined to a point P on the major arc of the circle, calculate the angle APB.

(6) Two chords PA and PB of a circle lie on opposite sides of the radius OP. Angle $OPA = 30°$ and angle $OPB = 20°$. If the points A and B are joined calculate the angles PAB and PBA.

(*Note.* Draw a tangent to the circle at P.)

(7) In a circle of radius 3 in. a chord AB is drawn. This chord is produced to a point P such that $AP = 8$ in. If the distance $BP = 4$ in. calculate the distance of P from the centre of the circle.

(8) *By a graphical construction* divide a line of length 5 in. into 7 equal parts.

(9) A man makes a rough measurement of the lengths of the sides of a field. If all his measurements of length are 5 per cent too big what percentage error in the area of the field will he make if he calculates it from these measurements?

(*It may be assumed that the measurements of the angles are correct.*)

(10) In what ratio must the sides of a polygon be reduced if its shape is to be maintained the same while its area is reduced to $\frac{2}{3}$ of its original area?

(11) The vertex, O, of a pyramid of wood is 24 in. vertically above the centre, K, of its base, which is a square of side 14 in. in a horizontal plane. Find—

(a) The volume of the pyramid;

(b) The total surface area, including base;

(c) The volume remaining when the top portion is removed by a saw-cut parallel to the base and bisecting OK. (*E.M.E.U.*)

(12) An open belt passes round two pulleys of diameter 3 ft. and 1 ft. respectively. If the centres of the pulleys are 4 ft. apart find the total length of the belt. (*U.E.I.*)

(13) A map is drawn to the scale of 6 in. to the mile. Express in acres the area represented on the map by a square of 2 in. side.

Given 640 acres = 1 sq. mile. What is the side of a square on the map whose area represents 90 acres? (*U.E.I.*)

(14) A steel pin is to be driven into a hole whose cross-section is the major segment of a circle of diameter 1 in. bounded by a chord of length 0·6 in. To make the pin, a 2-in. length of steel rod 1 in. in diameter was cut off, and a "flat" ground on it. Find the volume of metal removed by grinding. (*U.L.C.I.*)

(15) The following table gives related values of the flux density B and the magnetizing force H. Two sets of values of H are given, one set for H increasing the other for H decreasing.

B	0	1,000	3,000	5,000	7,000	8,500	9,500	10.000	10,500	10,800	11,000
H increasing	1·7	1·8	2	2·5	3·3	4·3	5·1	5·6	6·2	6·5	6·8
H decreasing	−2·1	−2	−1·7	−1·4	−0·9	0	1·1	2·2	3·7	5	6·8

Plot the curve connecting B and H for the above range of values, and determine the area enclosed by the curve and the axis of H. Express your answer in sq. in.

(*Note.* Use the following scales: B, 1 in. = 2000 units; H, 1 in. = 2 units.) (*U.L.C.I.*)

(16) The area of cross-section of a solid, by a plane perpendicular to an axis of the solid at a distance x in. from one end of the axis, is A sq. in. Values of A and x are given in the following table. The total length of the solid, along the axis, is 70 in Plot A against x and estimate the volume of the solid.

x	0	10	20	30	35	40	50	60	65	70
A	7	23 5	44	63	70	75	74 5	56	38 5	14

(*N.C.*)

(17) Express in terms of L, D and d (i) the volume, (ii) the total curved surface area (i e the area of the inside and outside surfaces, but not of the ends) of a cylindrical pipe whose length $= L$ ft ; external diameter $= D$ in.; internal diameter $= d$ in. What will be the effect of halving the values of D and d on (i) the volume, (ii) the area of the curved surface of the pipe? (*N.C.*)

(18) The speed of a train after leaving a station was—

Time, t sec. . . .	15	30	45	60	75	90
Speed (in miles per hour) .	5	12	30	40	55	65

Plot the speed v *in feet per second* against the time t sec

Divide the time into intervals δt, each 15 sec. Mark on the graph the product of $v\,\delta t$ for each interval. Write the usual expression for the sum of the products, and find its numerical value What does the answer mean in relation to the train? (*E M.E.U.*)

(19) Establish an expression for the area of a segment of a circle if the radius of the circle is r and the angle subtended at the centre is θ.

Calculate the cross-sectional area of the steam space of a Lancashire boiler 8 ft. in diameter if the water level is 5 ft from the bottom. (*U.E.I.*)

(20) The side of a trough is in the shape of a trapezium $ABCD$. A is the lowest corner, the bottom edge AB is inclined at 10° to the horizontal, the edges AD and BC are vertical, and the top DC is horizontal. If $DC = 20$ ft., $BC = 1$ ft, find the length of AB and the area of $ABCD$ (*N.C.*)

(21) Three equal circular discs, each of diameter 3 in, are placed on a flat table so that each disc touches the other two. Calculate the area of the uncovered part of the surface of the table enclosed by the discs (*N.C.*)

CHAPTER V

ALGEBRAIC GRAPHS

WE have already discussed in Chapter VII, Book I, the equations which lead to straight-line graphs. We also gave the fundamental equations of the circle and of the parabola. As these properties are of the greatest importance we give a summary of the results in tabular form and some further exercises on them. The student must fully master these, referring back where necessary before he proceeds to the new work.

1. The equation of the first degree gives a straight-line graph.

2. The line $x = a$, where a is a number, is a line parallel to the y axis YOY'; the line $y = b$, where b is a number, is a line parallel to the x axis XOX'.

3. If the equation of the first degree is arranged in the form $y = mx + c$, m is the increase of y per unit increase of x. It is called the *slope of the line.*

4. The equation of the straight line through the point whose co-ordinates are (x_1, y_1) and whose slope is m is

$$y - y_1 = m(x - x_1).$$

5. The distance between two points (x_1, y_1) (x_2, y_2) is $\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$.

6. The equation of the circle radius r and centre (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 = r^2$.

7. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ (coefficients of x^2 and y^2 are unity and there is no xy term) represents a circle. It can be written $(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$, so that the centre has co-ordinates $(-g, -f)$ and the radius is $\sqrt{\{g^2 + f^2 - c\}}$.

For the properties 5, 6, and 7 the scales for x and y must be equal; in questions dealing with these properties it will be assumed that they are equal without further mention of the point: they are *not* true if the scales are unequal.

8. The relation $y = x^2$ gives the fundamental equation of the parabola. The graph has the shape of a U which touches the x axis at the origin.

EXAMPLES XXVIII

(1) Calculate the equations of the lines
(i) through (3, 7) with slope 2,
(ii) through (– 3, 4) with slope 1,
(iii) through (2, – 4) with slope – 3,
(iv) through (– 2, – 1) with slope $\frac{1}{2}$.

(2) Write down the slopes of the lines
(i) $y = 2x - 7$, (ii) $2y = x + 5$,
(iii) $y + 3x = 4$, (iv) $2x + 5y + 6 = 0$.

(3) Calculate the slopes of the lines joining the points
(i) (8, 6) (9, 8), (ii) (– 3, 4) (5, 7),
(iii) (– 2, – 4) (0, 0), (iv) (3, – 4) (6, 0).
and hence find the equations of the lines.

(4) Calculate the slopes of the lines joining the points
(i) (5, 3) (2, 7) (ii) (– 3, 6) (1, 4),
(iii) (– 2, 1) (0, 0), (iv) (3, – 4) (2, 0),
and hence find the equations of the lines.

(5) Find the distances between the pairs of points
(i) (6, 3) and (9, 7),
(ii) (3, – 2) and (8, 10),
(iii) (16, 4) and (– 8, – 3).

(6) Obtain the equation of the circle whose centre is (3, – 2) and which passes through the point (8, 10)

(7) Obtain the co-ordinates of the centres and the radii of the circles
(i) $x^2 + y^2 + 4x - 6y - 3 = 0$. (ii) $4x^2 + 4y^2 - 12x - 10y = 27$.

(8) Find the equation of the circle whose centre is the point (4, 3) and whose radius is 7.

(9) Write down the co-ordinates of the centre of the circle of radius 3 which lies in the first quadrant and which touches the x axis at the point (5, 0).

(10) Find the co-ordinates of the centre of the circle of radius 5 which lies in the first quadrant and which cuts the x axis at the points (3, 0) and (9, 0).

(11) Find the equation of the circle of radius 7 lying in the third quadrant which touches both the x and y axes.

(12) Find the equations of the two circles which pass through the point (8, 1) and which touch both the x and y axes.

The Parabola, $y = ax^2 + bx + c$. We have seen already that the graph of $y = x^2$ is of the form of a U: the graph of $y = ax^2 + bx + c$ is of a similar general form but displaced and, possibly, inverted. In $y = x^2$ the lowest point of the loop is at the origin (0, 0) and is called the *vertex*, and the line $x = 0$ through it is called the *axis*; any two values of x numerically equal but of opposite sign (+ 2 and – 2, for example) give the same value for y, or, in other words, the curve is *symmetrical* about the axis. Where we are plotting the more general parabola we always start by calculating the co-ordinates of the vertex since—

(1) The position of the vertex gives us a general idea of the

position of the curve—if the vertex were at (20, 4), for example, it would be useless to start by calculating the values of y for $x = 1$, 2 or 3; as these values will be beyond the range of probable interest.

(2) Without exact knowledge of the position of the vertex we cannot "round off" the curve with any certainty when sketching in.

(3) The property of symmetry about the axis gives an immediate check on accuracy; thus, for example, if the vertex is at (20, 4) and we have calculated the values of y for $x = 17$ and $x = 23$ to be 10 and 12 respectively, we know at once

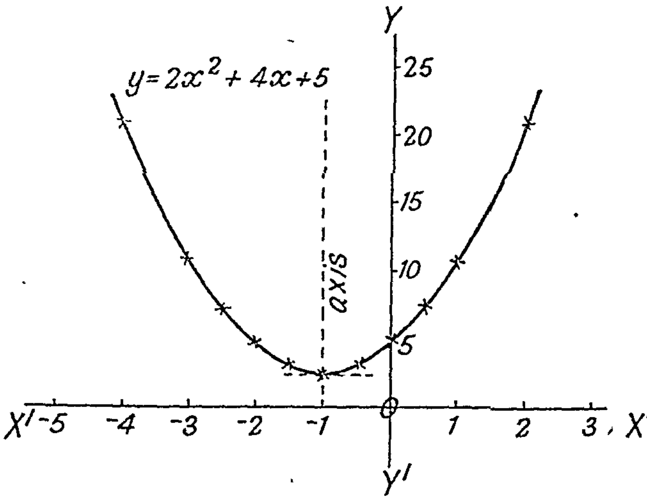


FIG. 61

that a slip has been made, since these values of y should be equal.

EXAMPLE 1. Plot the graph of $y = 2x^2 + 4x + 5$ (see Fig. 61).

We first find the vertex thus

$$\begin{aligned} y &= 2x^2 + 4x + 5 \\ &= 2(x^2 + 2x) + 5 \\ &= 2(x^2 + 2x + 1) + 5 - 2 \end{aligned}$$

(Completing the square within the bracket by adding 1, and taking away $2 \times 1 = 2$ outside the bracket to leave the right-hand side unchanged.)

or $$y - 3 = 2(x + 1)^2.$$

For the parabola $y = x^2$ we know that the vertex is given

by $x = 0$ and $y = 0$; comparing our new equation with $y = x^2$ we see, therefore, that the vertex is given by

$$x + 1 = 0 \text{ or } x = -1$$

and $$y - 3 = 0 \text{ or } y = 3,$$

or the vertex is the point $(-1, 3)$.

The effect of the multiplier 2 is to make the loop sharper than in a standard curve drawn to the same scale. We now make a table of values; as the x of the vertex is -1 it will be reasonable to calculate the y's from -4 to 2. We set the calculation out in tabular form.

x	-4	-3	-2	-1	0	1	2
$2x^2$ $+ 4x$ $+ 5$	32 -16 5	18 -12 5	8 -8 5	2 -4 5	0 0 5	2 4 5	8 8 5
y	21	11	5	3	5	11	21

(*Note.* The table is constructed by columns: thus, taking the column headed by -3, $2x^2 = 2(-3)^2 = 18$, $4x = 4(-3) = -12$, $5 = 5$ and, adding, $y = 11$. The calculations should always be set out in tabular form.)

We now plot the points from the table, but in an attempt to sketch the graph from these points we shall find it a little difficult to determine the curve near the vertex. We therefore calculate the y's for the additional pairs of points $x = -\frac{1}{2}$ or $-1\frac{1}{2}$ and $x = \frac{1}{2}$ or $-2\frac{1}{2}$. The student will notice that in the calculation for *additional points* we have used the property of symmetry about the axis ($x = -1$), using, in our calculation of y, the more convenient value of x in a pair, since the value of y is the same for both. Any slip in calculation will now be shown on plotting by a point definitely off the general run of the curve.

x	$-\frac{1}{2}$	$-1\frac{1}{2}$	$\frac{1}{2}$	$-2\frac{1}{2}$
$2x^2$ $+ 4x$ $+ 5$	$\frac{1}{2}$ -2 5		$\frac{1}{2}$ 2 5	
y	$3\frac{1}{2}$	$3\frac{1}{2}$	$7\frac{1}{2}$	$7\frac{1}{2}$

EXAMPLE 2. Plot the graph of $7y = 2 + 12x - 3x^2$ from $x = -1$ to $x = 5$ (see Fig. 62).

$$\begin{aligned} 7y &= 2 + 12x - 3x^2 \\ &= -3(x^2 - 4x) + 2 \\ &= -3(x^2 - 4x + 4) + 2 + 12, \end{aligned}$$

or

$$\begin{aligned} 7y - 14 &= -3(x-2)^2 \\ y - 2 &= -\tfrac{3}{7}(x-2)^2. \end{aligned}$$

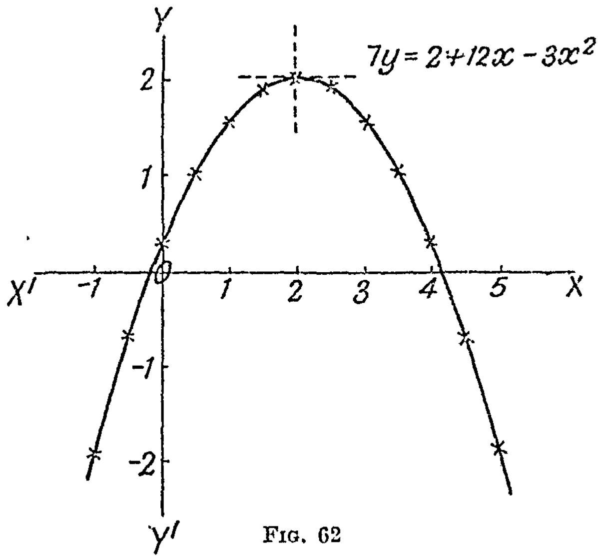


FIG. 62

The vertex is therefore the point (2, 2); the effect of the $\frac{3}{7}$ is to make the loop wider and that of the minus sign is to invert it.

	x	-1	0	1	2	3	4	5
	2	2	2	2	2	2	2	2
	$12x$	-12	0	12	24	36	48	60
	$-3x^2$	-3	0	-3	-12	-27	-48	-75
adding,	$7y$	-13	2	11	14	11	2	-13
	y	$-1\frac{6}{7}$	$\frac{2}{7}$	$1\frac{4}{7}$	2	$1\frac{4}{7}$	$\frac{2}{7}$	$-1\frac{6}{7}$

Plotting, we shall find it advisable to obtain y at the "half" points: it will be sufficient to calculate it for $x = -\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}$, and the others follow by symmetry.

x	$-\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$
2 $12x$ $-3x^2$	2 -6 $-\frac{3}{4}$	2 6 $-\frac{3}{4}$	2 18 $-6\frac{3}{4}$
$7y$ y	$-4\frac{3}{4}$ $-\frac{19}{28}$	$7\frac{1}{4}$ $1\frac{1}{28}$	$13\frac{1}{4}$ $1\frac{25}{28}$

and, by symmetry

x	$4\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$
y	$-\frac{19}{28}$	$1\frac{1}{28}$	$1\frac{25}{28}$

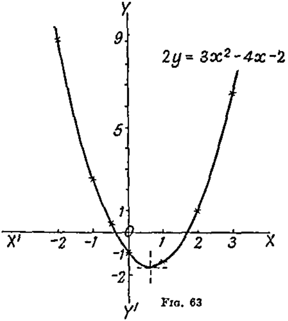


FIG. 63

EXAMPLE 3. Plot $2y = 3x^2 - 4x - 2$ from $x = -2$ to $x = 3$ (see Fig. 63).

$$\begin{aligned} 2y &= 3x^2 - 4x - 2 \\ &= 3\left(x^2 - \tfrac{4}{3}x\right) - 2 \\ &= 3\left(x^2 - \tfrac{4}{3}x + \tfrac{4}{9}\right) - 2 - \tfrac{4}{3}, \end{aligned}$$

or

$$\begin{aligned} 2y + \tfrac{10}{3} &= 3\left(x - \tfrac{2}{3}\right)^2 \\ y + \tfrac{5}{3} &= \tfrac{3}{2}\left(x - \tfrac{2}{3}\right)^2, \end{aligned}$$

and the vertex is the point $\left(\frac{2}{3}, -\frac{5}{3}\right)$.

x	-2	-1	0	1	2	3
$3x^2$	12	3	0	3	12	27
$-4x$	8	4	0	-4	-8	-12
-2	-2	-2	-2	-2	-2	-2
adding, $2y$	18	5	-2	-3	2	13
y	9	$2\frac{1}{2}$	-1	$-1\frac{1}{2}$	1	$6\frac{1}{2}$

For additional points the values for $x = -\frac{1}{2}$ and $x = 1\frac{1}{2}$ will be found sufficient.

x	$-\frac{1}{2}$	$1\frac{1}{2}$
$3x^2$	$\frac{3}{4}$	$6\frac{3}{4}$
$-4x$	2	-6
-2	-2	-2
$2y$	$\frac{3}{4}$	$-1\frac{1}{4}$
y	$\frac{3}{8}$	$-\frac{5}{8}$

In Examples 2 and 3 the graphs cut the x axis XOX', and the values of the x co-ordinates at these points are those for which y is zero; thus—

In Example 2, $y = 0$ when $x = 4{\cdot}16$ and when $x = -{\cdot}16$, or the roots of the equation $3x^2 - 12x - 2 = 0$ are $4{\cdot}16$ and $-{\cdot}16$ approximately.

In Example 3, $y = 0$ when $x = 1{\cdot}72$ and when $x = -{\cdot}39$, or the roots of the equation $3x^2 - 4x - 2 = 0$ are $1{\cdot}72$ and $-{\cdot}39$ approximately.

We gave in Book I a method by which the roots of a quadratic equation can be found by finding the intersections of the standard parabola $y = x^2$ with a straight line (in these cases the lines would be $3y = 12x + 2$ and $3y = 4x + 2$). The method now given is more accurate, but suffers the disadvantage that a new curved graph has to be drawn for each equation.

If the graph touches the axis XOX' the roots are equal; if (as in Example 1) the graph neither cuts the axis, nor touches it, then the equation has no roots in the ordinary meaning. In such a case a solution by formula leads to the square root of a negative quantity which has no real meaning, and the equation is said to have *imaginary roots*. (Thus from Example 1, if $2x^2 + 4x + 5 = 0$, we have by the formula

$x = \{-4 \pm \sqrt{(16-40)}\}/4 = \{-4 \pm \sqrt{-26}\}/4$; we cannot evaluate $\sqrt{-26}$ but $\{-4 + \sqrt{-26}\}/4$ and $\{-4 - \sqrt{-26}\}/4$ are called the imaginary roots of the equation.)

EXAMPLES XXIX

In Questions 1 to 4 draw the graphs of the given equations

(1) $y = x^2 - 6x + 8$ from $x = 0$ to $x = 6$.
(2) $y = x^2 - 2x + 3$ from $x = -1$ to $x = 4$.
(3) $4y = 4x^2 + 4x + 5$ from $x = -3$ to $x = 3$.
(4) $4y = 3 - 4x - 4x^2$ from $x = -3$ to $x = 3$.
(5) Solve graphically the equation $2x^2 - 3x - 3 = 0$.
(6) Solve graphically the equations $2y = 3x^2 + 6x + 2$, $2x - y + 2 = 0$.
(7) Solve graphically the equations $y = 2x^2 + 3x + 4$, $x + 2y = 9$.
(8) Solve graphically the equations $y = 5 + 2x - x^2$, $2y = x - 2$.
(9) Solve graphically the equations $y = 1 + 4x - 4x^2$, $x + 2y = 1$.
(10) Solve graphically the equations $y = -x^2 - 4x - 1$, $y = 1 + x$.

The Rectangular Hyperbola $(x - a)(y - b) = c$. In the case of the parabolas given in the preceding section we saw that the general shape followed that of one standard graph. The same applies to the group of graphs given by equations $(x - a)(y - b) = c$, and we shall take in detail the graph of $xy = 1$ and then consider the displacements and distortions caused by the constants a, b and c. These considerations in no way remove the necessity for calculations in a numerical case, but, exactly as in the parabola, some knowledge of the general run of the graph is of considerable help in the choice of convenient numbers for calculation and in the drawing of a smooth curve through the points.

$xy = 1$ (see Fig. 64). Graphing $y = \dfrac{1}{x}$, we first form a table of values for y for a series of values of x.

x	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4
y	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

x	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	-4
y	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$

We plot these points on squared paper, and find it advisable

to form an additional table in order to ensure the accuracy of our curve—

x	$\frac{3}{2}$	$\frac{2}{3}$
y	$\frac{2}{3}$	$\frac{3}{2}$

with similar negative values.

We notice the curve consists of two parts lying entirely in the first and third quadrants. If we were to take very large values of x (positive or negative) the corresponding values of

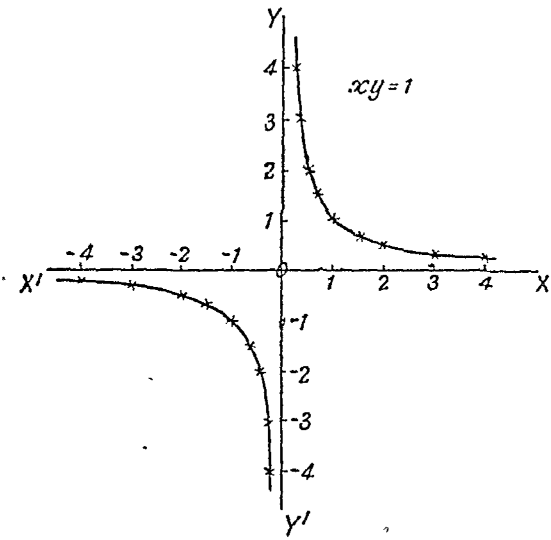


FIG. 64

y are very small and the curve would approach very closely to the x axis; if we were to take very small values of x the values of y would be very large, and the curve would approach very closely to the y axis. The lines, in this case $y = 0$ and $x = 0$, which the graph gradually approaches are called *the asymptotes.*

Note. We have tabulated values for x from $x = 4$ to $x = -4$, and, loosely, we say that we have drawn the graph from $x = 4$ to $x = -4$. The phrase is convenient but it is not strictly correct, as we must, of necessity, omit the section in which x is small.

We will now consider by means of examples some simple developments from the graph of $xy = 1$.

EXAMPLE 1. $xy + 4 = 0$, or $xy = -4$.

The effect of the 4 is to increase the size of the curve (i.e.

for any value of x or y it is further from the asymptotes $y = 0$ or $x = 0$ respectively; descriptively we might say "it is not so far in the corner"). If we give x any positive value the sign of y is negative and *vice versa*; the effect of the minus sign in -4 in the second form of the equation is to throw the curve into the second and fourth quadrants.

EXAMPLE 2. $(x + 2)(y - 1) = 2$.

Comparing with $xy = 1$, the line $x + 2 = 0$ clearly takes

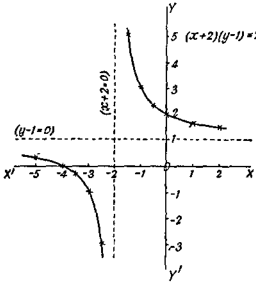


FIG. 65

the place of $x = 0$, and $y - 1 = 0$ the place of $y = 0$—the curve is displaced so that its asymptotes are $x + 2 = 0$ and $y - 1 = 0$: from the 2 the curve is "not so far in the corner."

From the equation we have $y = 1 + \dfrac{2}{x + 2}$ and tabulating y against x we have

x	-5	-4	-3	-1	0	1	2
y	$\frac{1}{3}$	0	-1	3	2	$1\frac{2}{3}$	$1\frac{1}{2}$

The student will realize that it is useless to attempt tabulation for the value $x = -2$ which makes y infinite; we take a series of values of x on each side of -2.

Plotting the points from the table it will be found advisable to obtain the additional points

x	$-3\frac{1}{2}$	$-2\frac{1}{2}$	$-\frac{1}{2}$
y	$-\frac{1}{3}$	-3	$2\frac{1}{3}$

and we can now sketch the curve which is given in Fig. 65.

EXAMPLE 3. $(x-2)(y+2)+1=0$, or $(x-2)(y+2)=-1$.

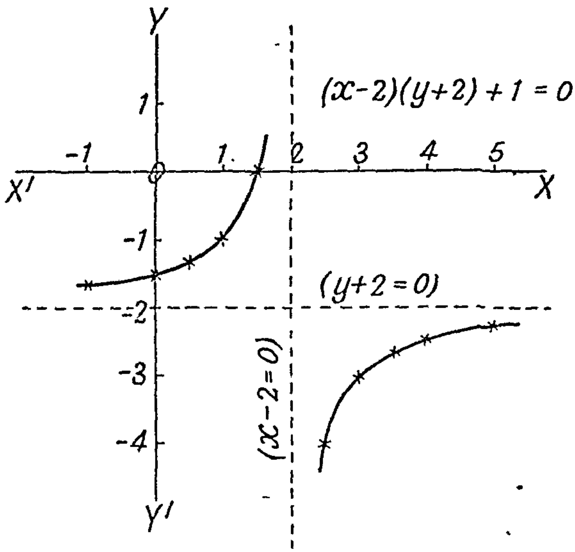


FIG. 66

The asymptotes are $x-2=0$ and $y+2=0$; the minus sign in the second form of the equation throws the graph into the displaced second and fourth quadrants. Expressing y in terms of x we have $y=-2-\frac{1}{x-2}$; in tabulating values of y against x, we avoid $x=2$, which makes y infinite, and take a series of values on each side of it.

x	-1	0	1	3	4	5
y	$-1\frac{2}{3}$	$-1\frac{1}{2}$	-1	-3	$-2\frac{1}{2}$	$-2\frac{1}{3}$

On plotting these points additional values will be found advisable—

x	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$
y	$-1\frac{1}{2}$	0	-4	$-2\frac{3}{4}$

and we can now sketch the graph as given in Fig. 66.

The graphs of the most general quadratic relation between x and y, which involves squared terms, are beyond the scope of this book, but we will illustrate two types which are of some practical importance by means of examples.

EXAMPLE 4. $4x^2 - 8xy + 9y^2 = 9$.

Plotting y against x we first calculate values—

x	$9y^2 - 8xy + 4x^2 - 9 = 0$	y
0	$9y^2 - 9 = 0$	1 or -1
$\frac{1}{2}$	$9y^2 - 4y - 8 = 0$	1·19 or $-$·75
1	$9y^2 - 8y - 5 = 0$	1·31 or $-$·42
$1\frac{1}{2}$	$9y^2 - 12y = 0$	0 or $1\frac{1}{3}$
2	$9y^2 - 16y + 7 = 0$	1 or $\frac{7}{9}$
$2\frac{1}{2}$	$9y^2 - 20y + 16 = 0$	no roots

x	$9y^2 - 8xy + 4x^2 - 9 = 0$	y
$-\frac{1}{2}$	$9y^2 + 4y - 8 = 0$	-1·19 or + ·75
-1	$9y^2 + 8y - 5 = 0$	-1·31 or ·42
$-1\frac{1}{2}$	$9y^2 + 12y = 0$	0 or $-1\frac{1}{3}$
-2	$9y^2 + 16y + 7 = 0$	-1 or $-\frac{7}{9}$
$-2\frac{1}{2}$	$9y^2 + 20y + 16 = 0$	no roots

The change of sign in the value of x (in the second column of values) changes the sign of y in the quadratic equation, and the roots can be written down from those in the first column, without the labour of calculation, merely by changes of sign: *this holds when x and y enter in the equation in x^2, xy, and y^2 terms only.*

The curve can now be drawn from the values obtained; it is the oval (*a*) in Fig. 67, and is known as an *ellipse*. We have already given in Book I, page 119 and Fig. 31, an example of

an ellipse; the student will recall that OB (half the major axis) was of length a and that OC (half the minor axis) was of length b. Taking OB and OC as OX and OY the corresponding equation is $b^2x^2 + a^2y^2 = a^2b^2$.

EXAMPLE 5. $2y = 3x + \dfrac{3}{4x}$.

The equation can be written $x(2y - 3x) = \frac{3}{4}$; comparing this form with $xy = 1$, the graph is a hyperbola with asymptotes

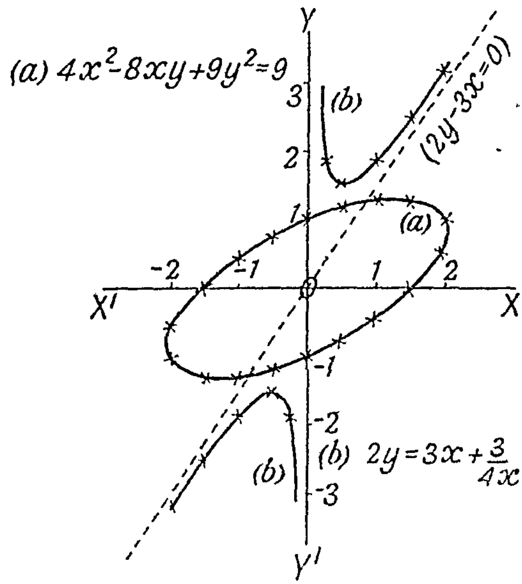


FIG. 67

$x = 0$ and $2y - 3x = 0$. It is, however, *not* a rectangular hyperbola, as the word rectangular is restricted to the case when the asymptotes are at right angles.

We form the table of values

x	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y	$1\frac{1}{2}$	$1\frac{7}{8}$	$2\frac{1}{2}$	$3\frac{3}{16}$

We notice that a change of sign of x merely changes the sign of y and, on plotting the points, find that with the

additional value $x = \frac{1}{2}$, $y = 1\frac{1}{2}$ we have enough to sketch the curve which is given by the *two* parts (b) in Fig. 67.

Note 1. The drawing of the asymptote is not an essential part of the plotting: it is only done as a matter of convenience.

Note 2. In both Examples 4 and 5 one set of values can be obtained from another merely by changing signs, and the curve is then centred on the origin. The student must not assume that this is always the case Thus, making slight alterations in the examples, it does not hold for $4x^2 - 8xy + 9y^2 + x = 9$ or $2y = 3x + \dfrac{3}{4x+1}$—it is, in fact, the exception, but where it does occur advantage should be taken of it to avoid unnecessary labour.

Notation. There is considerable variety in the wording used in questions on graphs, and in order that the student may be familiar with the common forms, variations have been introduced into the sets of examples; thus, instead of saying "draw the *graph* of $2y = 3x + \dfrac{3}{4x}$" we might say "draw the *curve* $2y = 3x + \dfrac{3}{4x}$" or "draw the graph of the *function* $\dfrac{3x}{2} + \dfrac{3}{8x}$." (The word "function" is merely an alternative word to "expression"; when we speak of a function of x we mean an algebraic expression in terms of x.) In some questions the range through which the graph is to be drawn is stated; in others it is not so, and the student is expected to draw it over a range wide enough to show the type of the curve, or, if the question is some problem on the use of graphs (as will be illustrated in a later section), to draw sufficient to meet the needs of the question. In some problems the student will be required to find the intersections of two graphs (for example, for the solution of the quadratic by means of the parabola $y = x^2$ and a straight line as given in Book I, Chapter VII); if the instruction to draw the graphs with the same axes and scales is omitted it should be understood as being implied.

The scales chosen should be large enough to give a well-balanced graph from which readings can easily be taken; they should not, however, be so large that the drawing of the graph is made cumbersome—accuracy can be lost if a graph is drawn too large as well as if it is drawn too small. It is often helpful towards the choice of suitable scales to make a rough sketch on ordinary paper after the table of values has been calculated before proceeding to the use of graph paper.

EXAMPLES XXX

In the following questions draw the graphs of the given equations.

(1) $2xy = 5$ from $x = -4$ to $x = 4$.
(2) $(x-3)(y+1)+3 = 0$ from $x = -1$ to $x = 6$.
(3) $xy - 2x + 4y = 2$ from $x = -6$ to $x = 1$.
(4) $xy + 3x - 3y - 11 = 0$ from $x = 0$ to $x = 6$.
(5) $xy + 4x + 2y + 6 = 0$ from $x = -5$ to $x = 1$.
(6) $xy + 3x + y + 7 = 0$ from $x = -4$ to $x = 2$.
(7) $4x^2 - 9y^2 = 1$ from $x = -3$ to $x = 3$.
(8) $4y^2 - x^2 = 9$ from $x = -4$ to $x = 4$.
(9) $4x^2 + 9y^2 = 36$.
(10) $x^2 + 4y^2 = 25$.
(11) $9x^2 - 8xy + 16y^2 = 36$.
(12) $x^2 + 6xy + 4y^2 = 9$.
(13) $y = 2x + \frac{6}{x}$ from $x = -3$ to $x = 3$.
(14) $y = 3x - \frac{2}{x}$ from $x = -4$ to $x = 4$.

To economize space we shall omit formal statements when we are graphing a curve and merely give the table of calculations for the values of y for given x and the graph of the curve. When the table of values is divided by a double line the section to the left gives the values which it would be natural to calculate first, and that to the right gives the additional values which it would be found advisable to calculate after those of the first section are plotted.

The cubic curves $y = ax^3 + bx^2 + cx + d$. The graph given by an equation $y = ax^3 + bx^2 + cx + d$, where a, b, c and d stand for numbers, belongs to one of three types. The student should familiarize himself with the general run of the types by examining the figures of the three illustrative examples (Figs. 68, 69, and 70). In the case of the parabola we showed how the graph could be compared with the standard one $y = x^2$. In the case of the cubic curves there are three types which can be compared with three standards: I, $y = x^3$, II, $y = x(x-1)(x+1)$, and III, $y = x(x^2+1)$. The numerical reduction, of which we give examples at the end of the section, is, however, often very heavy and it is usually sufficient to distinguish them by actual drawing.* In the cases illustrated, I, II and III, the coefficient a is positive; if a is negative the graph is turned upside down and goes off at great distances in the second and fourth quadrants.

* There is an alternative method of distinguishing between the types, without reducing the equations to the standards, but the method is beyond the scope of this book.

1. $y = x^3$, Fig. 68.

(The cube-power curve: taking any x the y gives the cube of x; taking any y the x gives the cube root of y.)

x	0	1	2	3	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
y	0	1	8	27	$\frac{1}{8}$	$3\frac{3}{8}$	$15\frac{5}{8}$

If we change the sign of x we merely change the sign of y, so there is no need to construct a separate table of values. The

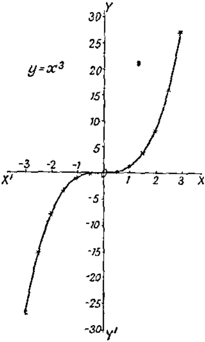


FIG. 68

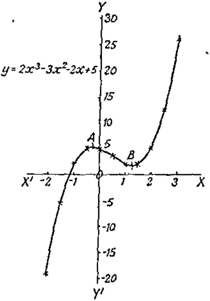


FIG. 69

figure gives the graph from $x = -3$ to $x = 3$. It will be noticed that for small values of x the graph is very flat, and for larger values of x it is very steep. It is important to realize that for values of x between -1 and 1 the graph, as drawn, is quite useless; if we were interested in this range we should have to draw this part of the graph on a very much larger scale, taking values of $x = \cdot 1, \cdot 2, \cdot 3$, etc.

We shall illustrate the second and third types by numerical examples drawn from $x = -2$ to $x = 3$.

II. $y = 2x^3 - 3x^2 - 2x + 5$, Fig. 69.

x	-2	-1	0	1	2	3	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$2x^3$	-16	-2	0	2	16	54	$-\frac{27}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{27}{4}$	$\frac{125}{4}$
$-3x^2$	-12	-3	0	-3	-12	-27	$-\frac{27}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{27}{4}$	$-\frac{75}{4}$
$-2x$	4	2	0	-2	-4	-6	3	1	-1	-3	-5
$+5$	5	5	5	5	5	5	5	5	5	5	5
y	-19	2	5	2	5	26	$-5\frac{1}{2}$	5	$3\frac{1}{2}$	2	$12\frac{1}{2}$

(*Note.* The points A and B shown in the figure are for use in a later example.)

The graph has a double bend. The example given in the figure cuts the x axis in one point only. Others of the same type may cut in three points, as, for example,

$$y = (x-1)(x-2)(x-3)$$
$$= x^3 - 6x^2 + 11x - 6$$

which cuts the axis of x where $x = 1$, 2 and 3; or cut in one point and touch in another as does

$$y = (x-1)(x-2)^2$$
$$= x^3 - 5x^2 + 8x - 4$$

which cuts the axis of x where $x = 1$ and touches where $x = 2$.

(If an equation is given in terms of factors, e.g.

$$y = (x-1)(x-2)(x-3),$$

the form is quite as convenient for numerical evaluation as the one in which the expression is multiplied out,

$$y = x^3 - 6x^2 + 11x - 6.)$$

III. $y = x^3 - 3x^2 + 5x + 11$, Fig. 70.

x	-2	-1	0	1	2	3	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
x^3	-8	-1	0	1	8	27	$-\frac{27}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{27}{8}$	$\frac{125}{8}$
$-3x^2$	-12	-3	0	-3	-12	-27	$-\frac{27}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{27}{4}$	$-\frac{75}{4}$
$+5x$	-10	-5	0	5	10	15	$-\frac{15}{2}$	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{15}{2}$	$\frac{25}{2}$
$+11$	11	11	11	11	11	11	11	11	11	11	11
y	-19	2	11	14	17	26	$-6\frac{5}{8}$	$7\frac{5}{8}$	$12\frac{7}{8}$	$15\frac{1}{8}$	$20\frac{3}{8}$

Curves of this type cut the axis of x in one point only (see

Fig. 70). It will be noticed both in the table of values for this example and in the tables for previous ones, that we have left the values in fractions where the numbers are familiar; when the scales are small, it is easier to plot, say, $\frac{1}{4}$ or $\frac{3}{8}$ than ·25 or ·375.

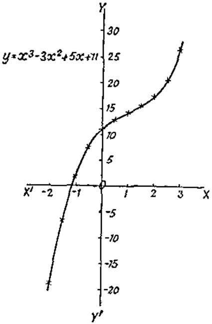


FIG. 70

The Numerical Reduction to Standard Form. The student has already become familiar with the process of "completing the square" in his work on the quadratic equation and on the graphing of the parabola. To reduce the cubic graphs to standard, we use the process of "completing the cube" by one or other of the formulae

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Thus—

(i) If $\qquad y = x^3 + 6x^2 + 12x + 9$

we have $\qquad y = (x+2)^3 + 9 - 8$

$$y - 1 = (x+2)^3.$$

and the curve is merely the standard curve moved two units to the left and one unit upwards ($x + 2$ takes the place of x and $y - 1$ takes the place of y; the part of the curve at O is moved to the point where $x + 2 = 0$, $y - 1 = 0$, or $x = -2$, $y = 1$).

(ii) If
$$y = 8x^3 - 36x^2 + 54x + 22$$
we have
$$y = (2x - 3)^3 + 22 - 27$$
$$y + 5 = 8\,(x - \tfrac{3}{2})^3,$$

and the curve is moved so that the part corresponding to that at O (the flat part) is at the point $\frac{3}{2}, -5$; the effect of the 8 is to make the curve more steep.

(iii) If
$$3y = 7 - 3x + 3x^2 - x^3$$
we have
$$3y = 6 - (x^3 - 3x^2 + 3x - 1)$$
$$3y - 6 = -(x - 1)^3$$
or
$$y - 2 = -\tfrac{1}{3}\,(x - 1)^3,$$

and the curve is moved so that the flat part is at the point 1, 2; the effect of the $\frac{1}{3}$ is to make the curve less steep and that of the minus sign in $-\frac{1}{3}$ is to turn the curve upside down.

(iv) If $y = 2x^3 - 3x^2 - 2x + 5$ (Fig. 69)

we have
$$\begin{aligned} y &= 2(x^3 - \tfrac{3}{2}x^2) - 2x + 5 \\ &= 2\,[x^3 - 3x^2\,(\tfrac{1}{2}) + 3x\,(\tfrac{1}{2})^2 - (\tfrac{1}{2})^3] \\ &\qquad\qquad - 2x - \frac{3x}{2} + 5 + \tfrac{1}{4} \\ &= 2\,(x - \tfrac{1}{2})^3 - \tfrac{7}{2}\,(x - \tfrac{1}{2}) + \tfrac{21}{4} - \tfrac{7}{4} \end{aligned}$$
or
$$\begin{aligned} y - \tfrac{7}{2} &= 2\,(x - \tfrac{1}{2})^2\,\{(x - \tfrac{1}{2})^2 - \tfrac{7}{4}\} \\ &= 2\,(x - \tfrac{1}{2})\left\{(x - \tfrac{1}{2}) - \frac{\sqrt{7}}{2}\right\}\left\{(x - \tfrac{1}{2}) + \frac{\sqrt{7}}{2}\right\}, \end{aligned}$$

and we have the standard type II with the "centre point" moved to $x = \frac{1}{2}$, $y = \frac{7}{2}$; the effects of the $\sqrt{7}/2$ and of the 2 are manifest.

(v) If $y = x^3 - 3x^2 + 5x + 11$ (Fig. 70)

we have
$$\begin{aligned} y &= (x^3 - 3x^2) + (5x + 11) \\ &= (x^3 - 3x^2 + 3x - 1) + 5x - 3x + 11 + 1 \\ &= (x - 1)^3 + 2(x - 1) + 14 \end{aligned}$$

or $y - 14 = (x - 1)\left\{(x - 1)^2 + 2\right\}$,

and we have the standard type III with the "centre point" moved to $x = 1$, $y = 14$.

EXAMPLES XXXI

(1) Draw the graph of $y = x^3$ from $x = -3$ to $x = 3$ and from it obtain approximately the values of $2{\cdot}3^3$, $(-1{\cdot}7)^3$, $\sqrt[3]{24}$, $\sqrt[3]{(-17.)}$

In Questions 2 to 10 plot the graphs of—

(2) $y = (x - 1)(x - 2)(x - 3)$ from $x = 0$ to $x = 4$.
(3) $y = (2x + 1)(x - 1)^2$ from $x = -2$ to $x = 2$.
(4) $y = x^3 - 6x + 7$ from $x = -3$ to $x = 3$.
(5) $y = 3 + 2x + x^2 - x^3$ from $x = -3$ to $x = 3$.
(6) $y = x^3 + 5x + 2$ from $x = -2\frac{1}{2}$ to $x = 2\frac{1}{2}$
(7) $y + x = x^2 + x^3$ from $x = -2\frac{1}{2}$ to $x = 2\frac{1}{2}$
(8) $y = x^3 - 6x^2 + 12x$ from $x = -1$ to $x = 5$.
(9) $10y = 11 - 6x + 12x^2 - 8x^3$ from $x = -3$ to $x = 3$.
(10) $y = 1 - 4x - 3x^2 - x^3$ from $x = -3$ to $x = 2$

The Logarithmic Curve $y = \log_{10}x$. When he is plotting the graph of $\log_{10}x$ the student must remember that the logarithm is defined for positive values of x only, so that the curve lies entirely to the right of the axis of y, YOY'. For values of x greater than 1 the values of y can be plotted straight from the readings in the logarithmic tables; this can also be done for values of x less than 1 by a very simple device. We use a double system for the marking of the axis of y; we first mark the graduations in the ordinary way $-{\cdot}1$, $-{\cdot}2$, . . . , -1, $-1{\cdot}1$, etc., and then we can re-mark them as $\bar{1}{\cdot}9$, $\bar{1}{\cdot}8$ . . . . $\bar{1}$, $\bar{2}{\cdot}9$, etc., since (for example) $\bar{1}{\cdot}8 = -1 + {\cdot}8 = -{\cdot}2$. If we have done this second marking and wish to plot the logarithm of ·3, we look up log 3 in the tables and obtain ·477. We are then able to plot at once log ·3 as $\bar{1}{\cdot}477$ without having to go through the intermediate stage $-1 + {\cdot}477 = -{\cdot}523$. The shape of the graph is given in Fig. 71.

(*Note.* It is not always necessary to enter the two types of marking, as the process can be done mentally—thus if 1 small square represents ·2 along the y axis and we want the position of $\bar{1}{\cdot}2$ all we have to do is to start at -1 and proceed a distance of 1 small square *upwards* from it.)

Because $\log_a x = \dfrac{\log_{10}x}{\log_{10}a}$ the graph of $y = \log_a x$, where a

is any base, is similar to that of $y = \log_{10}x$; in fact, if we write k for $\log_{10}a$, we have

if $$y = \log_a x \text{ then } ky = \log_{10}x,$$

and we can transform the logarithmic curve to the base 10

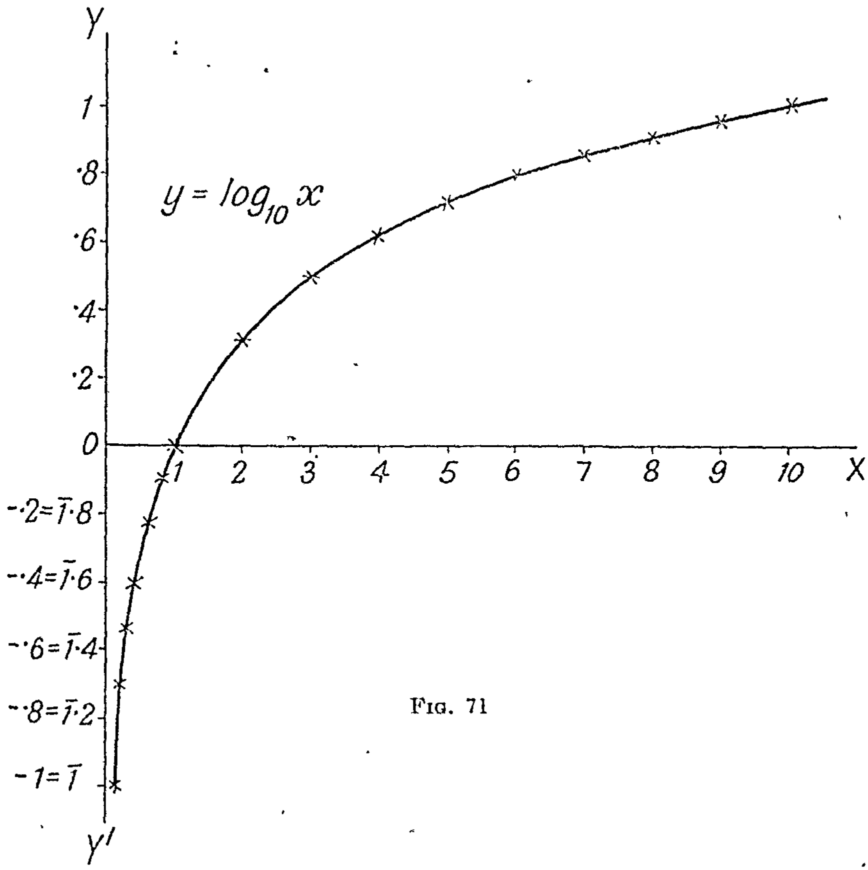


FIG. 71

immediately into that to base a by changing the graduations of Fig. 71 from 1, 2, etc., to $1/k$, $2/k$, etc. We cannot, however, retain the alternative "bar" notation for negative values of y. If, however, it is desired to plot $\log_a x$ directly without thinking of it first to base 10, all that is needed is to look up each reading in the log tables to base 10 and then to multiply it by $\log_a 10$ before the point is plotted, since $\log_a x$ also equals $\log_{10}x \times \log_a 10$.

The Power Curve $y = a^x$. If $y = a^x$ then $x = \log_a y$ and the

power curve is therefore identical with the logarithmic curve with the axes interchanged. Since $y = \text{antilog}_a x$ we might also call it an "antilog curve." The graphs for $a = 1, 2, 3, 4, \frac{1}{3}$, and ε ($= 2{\cdot}718$) have been drawn (see Fig. 72) from the table given. It will be noticed that all the curves cross at $x = 0$, $y = 1$ (since $a^0 = 1$ for all values of a), and that the curve for

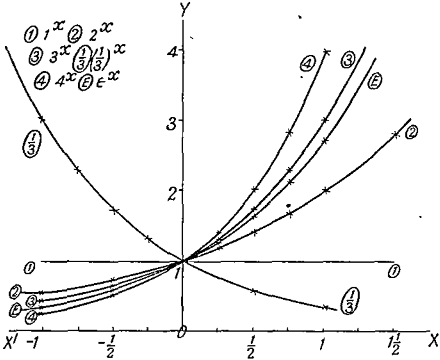


FIG. 72

$(\frac{1}{3})^x$ is that for 3^x reversed (since $(\frac{1}{3})^x = 3^{-x}$); a separate table has therefore not been constructed for this curve.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
2^x	·5	·71	1	1·19	1·41	1·68	2
3^x	·33	·58	1	1·32	1·73	2·28	3
4^x	·25	·5	1·	1·41	2	2·89	4
ε^x	·37	·61	1	1·28	1·65	2·12	2·72

For 1^x, $y = 1$ for all values of x, and the graph is the straight line ①.

Parametric Co-ordinates. It is sometimes convenient to express the values of x and y for a point on a curve by means of a third variable, which is called a *parameter*. The method is of some importance in the applications of Mathematics to Mechanics. The method of representing the co-ordinates of a point will be most clearly explained by means of an example.

EXAMPLE. Suppose a circle of radius a rolls along the x axis. It starts in the position shown in the figure by the dotted circle with the point P of the circumference at the origin O. Find

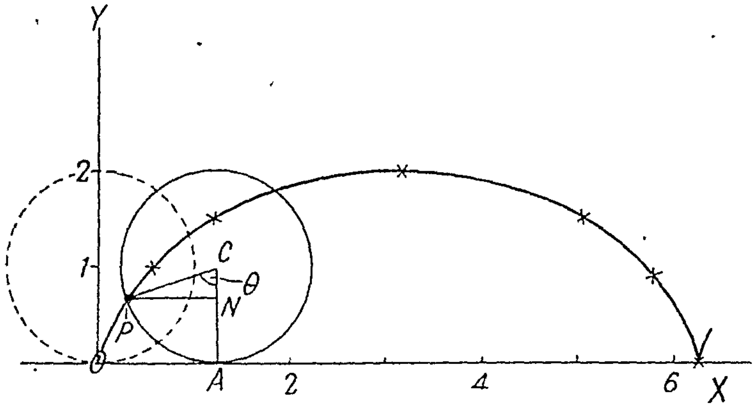


FIG. 73

the co-ordinates of P when the circle has rolled through an angle θ radians.

Let C be the centre of the circle and A the point of contact with the x axis (see Fig. 73, in which a is taken as 1); the angle PCA is θ and the length of the arc PA is $a\theta$. It follows that $OA = a\theta$ since the circle has rolled along the axis. Draw PN perpendicular to CA; then, as $CP = CA = a$, we have for the x and y co-ordinates of P

$$x = OA - PN = a\theta - a\sin\theta = a(\theta - \sin\theta)$$

$$y = AN = AC - CN = a - a\cos\theta = a(1 - \cos\theta).$$

The co-ordinates x and y are then said to be expressed in terms of the parameter θ as $a(\theta - \sin\theta)$ and $a(1 - \cos\theta)$ respectively.

The figure has been drawn for one complete turn of the circle (θ from 0 to 2π); the curve then repeats itself, and the commencement of the next section is shown.

θ	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	2π
	0	90°	135°	180°	225°	270°	360°
$x/a = \theta - \sin\theta$	0	·57	1·65	3·14	4·59	5·67	6·28
$y/a = 1 - \cos\theta$	0	1	1·71	2	1·71	1	0

This curve has been used in a device to improve the time-keeping qualities of the pendulum. The upper end of the pendulum was made flexible and suspended between guides whose shape was obtained by inverting the part of the curve near the "V." The device corrected errors due to the magnitude of the swing, but as these are negligible when compared with other errors it has not been extensively adopted.

Polar Co-ordinates. In this system the position of a point P (see Fig. 74) is determined by its distance r measured from a fixed point O called the pole, and the angle $\theta = X\hat{O}P$ is measured in the positive direction (i.e. counter-clockwise) from a fixed line OX through O; the polar co-ordinates of P are then said to be r, θ. If we take OX as the x axis, and draw OY at right angles to it $\left(\text{so that } XOY = \frac{\pi}{2}\right)$ as the y axis for ordinary x, y co-ordinates, then clearly

$$x = r\cos\theta,\ y = r\sin\theta.$$

If, in any given case, the angle θ is negative then the position of OP is obtained by a negative rotation; if r is negative we first determine from θ the position of the line OP and then measure a distance r *backwards* through O. In Fig. 74 the points are plotted as

Point	r	θ
P	2	$\frac{\pi}{3}$
Q	-1	$\frac{\pi}{4}$
R	1	$-\frac{\pi}{4}$
S	2	$-\frac{5\pi}{3}$

but equally well we could take as the co-ordinates of Q, R and S

Point	r	θ
Q	1	$\frac{5\pi}{4}$
R	1	$\frac{7\pi}{4}$
S	2	$\frac{2\pi}{3}$

The student should check these co-ordinates for himself on the figure.

EXAMPLE 1. The polar co-ordinates of N, the foot of the

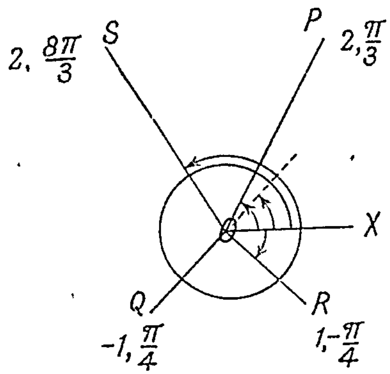


FIG. 74

perpendicular from O on to a straight line, are b, α. Find the polar equation of the line.

Let the co-ordinates of P, any point on the line, be r, θ (see Fig. 75 (i)). The triangle NOP is right-angled at N, $OP = r$, $ON = b$,

and $$N\hat{O}P = X\hat{O}P - X\hat{O}N = \theta - \alpha.$$

But $ON = OP\cos NOP$, and therefore, replacing the lengths and the angle by symbols, the equation of the line is

$$r \cos(\theta - \alpha) = b.$$

EXAMPLE 2. A circle of radius a passes through O; the polar co-ordinates of the centre C are a, α. Find the polar equation of the circle.

Let $P\ (r,\ \theta)$ be any point on the circle. Joining OC, PC, OCP is an isosceles triangle (Fig. 75 (ii)) in which

$$OC = PC = a$$

and $$C\hat{P}O = C\hat{O}P = X\hat{O}P - X\hat{O}C = \theta - \alpha.$$

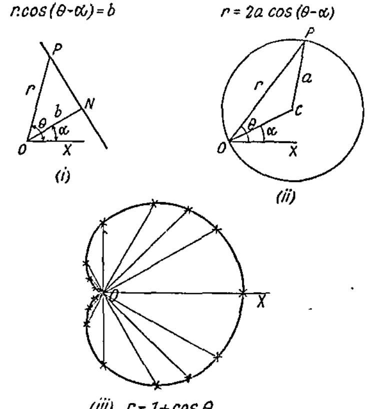


Fig. 75

. From the isosceles triangle $OP = 2OC\cos COP$, and the required equation is therefore

$$r = 2a \cos\ (\theta - \alpha).$$

EXAMPLE 3. Draw the curve $r = a\ (1 + \cos \theta)$.

For convenience we take a as 1 and form the table of values for r corresponding to values of θ from 0 to π (the same values hold for θ 2π to π) to plot the curve given in Fig. 75 (iii).

θ	0 0	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{3\pi}{4}$ 135°	$\frac{5\pi}{6}$ 150°	π 180°
r	2	1·87	1·71	1·50	1	·5	·29	·13	0

The curve, which is called the *cardioid*, from its resemblance to the popular idea of the shape of the heart, is of importance *in engineering work as it gives the form of the cam used in* many valve controls (the cam is pivoted at O; only the section of the curve away from O is generally used).

EXAMPLES XXXII

(1) Draw the curve $y = \log_{10}x$ from $x = \cdot 2$ to $x = 8$, and from it determine the logarithms of 6·7 and ·75 and the numbers whose logs are ·25, $\bar{1}\cdot 5$ and 1·41.

(2) Draw the curve $10y = 5^x$ from $x = -\frac{1}{2}$ to $x = 4$. From readings on the curve determine approximately the values of $5^{\frac{3}{4}}$, $5^{3\cdot 5}$ and the power of 5 which equals 55.

(3) Draw the graph of $y = \log_4 x$ from $x = \frac{1}{4}$ to $x = 10$, and from it determine the logarithms to base 4 of 6·5 and 0·65.

(4) Draw the graph of the curve given by $x = 4 + 3t$, $y = t^2$ from $t = -5$ to $t = 5$.

(5) Draw the graph of the curve given by $x = t + 1$, $y = \dfrac{2}{t}$ from $t = \frac{1}{4}$ to $t = 4$.

(6) Draw the graph of the curve given by $x = t + 2$, $y = t^3$ from $t = -3$ to $t = 3$.

(7) Find in terms of x and y the equation of the curve given by $x = 4t + 1$, $y = 3t^2$.

(8) Find in terms of x and y the equation of the curve given by $x = 5\cos\theta$, $y = 5\sin\theta$.

(9) Find in terms of x and y the equation of the curve given by $x = 4\cos\theta$, $y = 5\sin\theta$.

(10) Plot the points given in polar co-ordinates by $3, \dfrac{\pi}{4}$; $2, \dfrac{3\pi}{2}$; $-2, \dfrac{\pi}{2}$.

(11) Plot the points given in polar co-ordinates by $2, 0$; $-2, \pi$; $1, -\dfrac{\pi}{3}$; $-1, \dfrac{2\pi}{3}$.

(12) Draw the graph of the straight line $r\cos\left(\theta - \dfrac{\pi}{3}\right) = 2$.

(13) Draw the graph of the straight line $r\cos\left(\theta + \dfrac{\pi}{4}\right) = 3$.

(14) Draw the graph of the straight line $r\cos\left(\theta + \dfrac{\pi}{6}\right) = -1$.

(15) Draw the graph of the circle $r = 4\cos\left(\theta - \dfrac{\pi}{3}\right)$

(16) Draw the graph of the circle $r = -3\cos\left(\theta + \frac{\pi}{4}\right)$.

(17) Find the equation to the circle which has the normal from the pole to the straight line $r\cos\left(\theta - \frac{\pi}{6}\right) = 2$ as diameter.

(18) Find the equation to the tangent to the circle $r = 2\cos\left(\theta - \frac{\pi}{6}\right)$ at the point $\theta = \frac{\pi}{3}$

(19) Plot the graph of $r = 2 + \cos\theta$.

(20) Draw the graph of $r = 1 + 2\cos\theta$.

Further Graphical Applications

The Roots of an Equation. We have already seen in the case of the parabola that the values of x where the graph cuts the x axis XOX' give the roots of the quadratic equation in x obtained by putting y equal to 0. This property can clearly be extended to any graph, and we have—

The points where the graph of any relation between x and y cuts the x axis give the roots of the equation in x obtained by putting y equal to 0 in the relation.

Thus, taking an example we have already drawn—

$y = 2x^3 - 3x^2 - 2x + 5$ (Fig. 69) cuts the x axis in one point only; the equation $2x^3 - 3x^2 - 2x + 5 = 0$ is satisfied by only one value of x (approximately $-1{\cdot}13$).

(We can have as many intersections as the degree of the equation; if the graph touches the axis the point takes the place of two (or more) intersections, and we say the equation has two (or more) equal roots $y = (x-1)^3(x-2)^2$ meets $y = 0$ three times at $x = 1$ and twice at $x = 2$; $(x-1)^3(x-2)^2 = 0$ has three roots equal to 1 and two equal to 2. If the number of intersections, allowing for contacts, is less than the degree we say the curve has the remaining roots imaginary. In the example first quoted the equation is of degree 3 and has 1 real and 2 imaginary roots.)

The Points of Intersection of Two Graphs. Consider the points of intersection of any two graphs, say $y = x^2 - 2x$ and $y = \frac{2}{x}$. At a point of intersection the y's of the two have the same value for the *same* x—for the x, in fact, of the point of intersection; and the x co-ordinates of the points of intersection are therefore the roots of the equation

$$x^2 - 2x = \frac{2}{x}$$

or

$$x^3 - 2x^2 - 2 = 0.$$

By similar reasoning the y co-ordinates are the roots of the equation

$$\left(\frac{2}{y}\right)^2 - 2\left(\frac{2}{y}\right) = y$$

or
$$y^3 + 4y - 4 = 0.$$

Maximum and Minimum Values. If we examine the graph of $y = 2x^3 - 3x^2 - 2x + 5$, given in Fig. 69, and start from $x = -2$, we see that y increases until we get to A ($x = -\cdot 26$, $y = 5\cdot 3$) and then decreases to B ($x = 1\cdot 26$, $y = 1\cdot 7$), after which it increases again. At A the value of y is greater than its value at any point near A (any point includes points on both sides of A); y is said to have a *maximum* value at A. At B the value of y is less than its value at any point near B; y is said to have a *minimum* value at B. In other words, if at any point on the curve the tangent is parallel to the x axis XOX', the point gives a *maximum* y when the curve near it is entirely *below* the tangent and gives a *minimum* y when the curve near it is entirely *above* the tangent.

It is important that students should realize that "maximum" and "minimum" are not the same thing as "greatest" and "least." They can be the same (as in the parabola figures 61, 62, 63), but they are clearly not so in this case. Also the maximum may actually be less than the minimum, as in the case of the curve (*b*) in Fig. 67. If the curve crosses the tangent (as at $x = 0$, Fig. 68) the position is neither a maximum nor a minimum.

We will finish the chapter by giving two examples of graphical problems.

EXAMPLE 1. Draw the graph of $y = x(x - 1)(3 - 2x)$ from $x = -\cdot 2$ to $x = 2$, and find graphically the values of x for which

(i) $4x(x - 1)(3 - 2x)$ is a maximum or a minimum,
(ii) $4x(x - 1)(3 - 2x)$ equals $(x - 2)$,
(iii) $4x(x - 1)(3 - 2x)$ exceeds $(x - 2)$ by 1.

Calculating values of y for values of x, we have at sight $y = 0$ when $x = 0$, 1 or $1\cdot 5$ and also

x	$-\cdot 2$	$\cdot 2$	$\cdot 4$	$\cdot 6$	$\cdot 8$	$1\cdot 2$	$1\cdot 4$	$1\cdot 6$	$1\cdot 8$
$x - 1$ $3 - 2x$	$-1\cdot 2$ $3\cdot 4$	$-\cdot 8$ $2\cdot 6$	$-\cdot 6$ $2\cdot 2$	$-\cdot 4$ $1\cdot 8$	$-\cdot 2$ $1\cdot 4$	$\cdot 2$ $\cdot 6$	$\cdot 4$ $\cdot 2$	$\cdot 6$ $-\cdot 2$	$\cdot 8$ $-\cdot 6$
product y	$\cdot 816$	$-\cdot 416$	$-\cdot 528$	$-\cdot 432$	$-\cdot 224$	$\cdot 144$	$\cdot 112$	$-\cdot 192$	$-\cdot 864$

(i) Find by trial the places where the tangent is parallel to the axis of x; we have (see Fig. 76)—

the maximum is at A when $x = 1{\cdot}27$;
the minimum is at B when $x = {\cdot}39$.

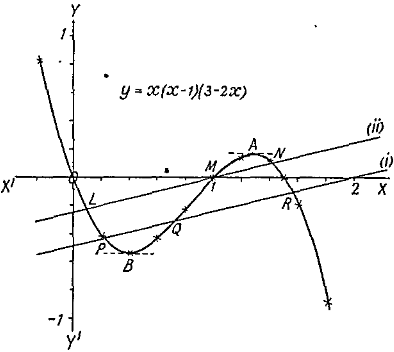


FIG. 76

(ii) Draw the graph of $y = \dfrac{x-2}{4}$ (line (i) in the figure); at the points P, Q and R where it cuts $y = x(x-1)(3-2x)$ the y's of the two graphs are the same for the same x's, or

$$4x(x-1)(3-2x) = (x-2) \text{ when } x = {\cdot}23,\ {\cdot}68,\ 1{\cdot}57.$$

(iii) Draw the graph of $y = \dfrac{x-2}{4} + \dfrac{1}{4}$; then, at the points L, M and N, $x(x-1)(3-2x)$ is greater than $\dfrac{x-2}{4}$ by $\dfrac{1}{4}$, or $4x(x-1)(3-2x)$ exceeds $(x-2)$ by 1. The values of x at these points are, approximately, $\cdot 1$, 1 and $1{\cdot}4$.

EXAMPLE 2. Solve graphically the equations

$$y = \log_{10} x^2$$
$$\cdot 7x + 1{\cdot}2y + \cdot 84 = 0,$$

given that x lies between -3 and 3.

It will be remembered that the graph of $\log_{10} x$ only exists when x is positive (see Fig. 71); x^2, however, is always positive

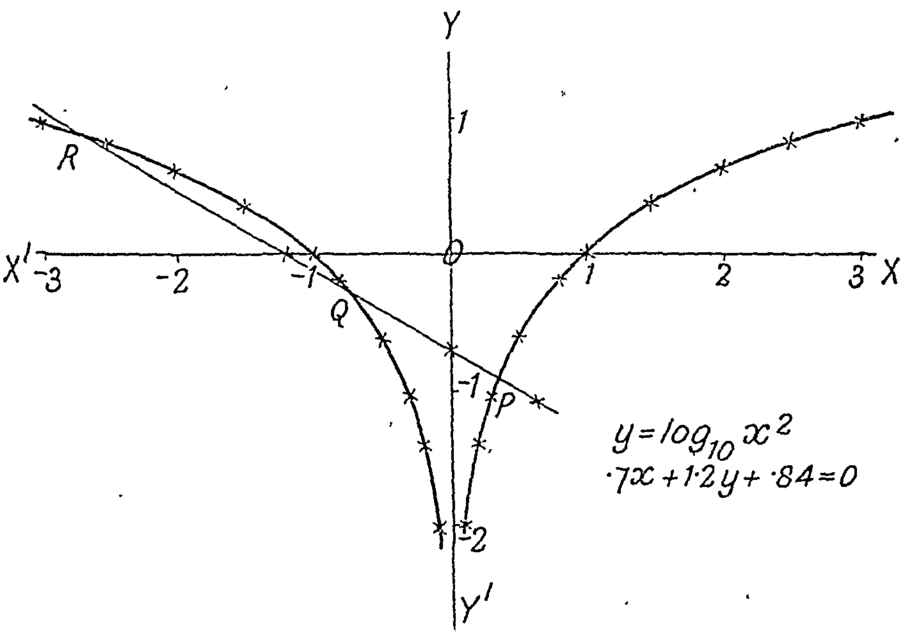


FIG. 77

and the graph of $\log x^2$ exists for positive and negative values of x. The easiest way of plotting this graph is to remember that $\log x^2 = 2 \log x$ for any positive value of x, so all we have to do is to look up the value of $\log x$ and double it for positive values of x up to 3, thus

$$x = 2, \quad \log x = \cdot 3010; \quad y = \log x^2 = \cdot 602.$$

This saves the working out of x^2. If we take numerically equal but negative values of x the value of x^2 is unchanged ($x = -2$, $y = \cdot 602$). To plot $\cdot 7x + 1{\cdot}2y + \cdot 84 = 0$ (a straight line) we take two points $x = 0$, $y = -\cdot 7$, and $y = 0$, $x = -1{\cdot}2$

with a check point $x = \cdot 6$, $y = -1\cdot 05$ The roots, which are given by the x and y co-ordinates of P, Q, R in Fig. 77 are approximately $x = \cdot 35$, $y = -\cdot 91$; $x = -\cdot 68$, $y = -\cdot 34$; $x = -2\cdot 75$, $y = \cdot 88$.

(Additional elementary examples on the work of this chapter will be found on page 178.)

EXAMPLES XXXIII

(1) Plot the graphs of $4y = x^2 + 2x - 3$ and $3y + x = 2$. Write down the x co-ordinates of the points of intersection, and from *the given equations* form the equation of which these should be the roots.

(2) Draw the graph of $y = 3x^2 - 13x + 7$. For what values of x is y equal to $x - 2$?

(3) Plot the graphs of the functions $(3x + 5)/4$ and $2x - x^2$ from $x = -1$ to $x = 3$. By a graphical construction determine approximately the values of x for which the first function exceeds the second by 2. (*Draw the graph of* $(3x + 5)/4 - 2$.)

(4) The costs of production of a factory come under three heads: a standing charge of £1,000 per annum, and variable costs of £$60x$ and £$(14x - x^2)20$, where x is the number of hundreds of tons of goods produced. With the same scales and axes draw curves to show the three sets of costs and from them construct a total costs curve from $x = 0$ to $x = 10$. Determine the output for which the cost is greatest.

(5) The equation $x^3 - 3x - 1 = 0$ may be solved by graphing $y = x^2$ and $y = 3 + \frac{1}{x}$. Graph these and hence find the roots of the equation roughly to one place of decimals.

(6) With the same scales and axes plot the graphs of $2y = x + \frac{1}{x}$ and $y = 2x - 2$ from $x = 3$ to $x = -3$. Read the x co-ordinates of the points of intersection.

(7) Plot the curves $y = x^2$, $2y + 3x = 6$, from $x = 0$ to $x = 2\frac{1}{2}$. Hence find a root of the equation $2x^2 + 3x - 6 = 0$ to one place of decimals

(8) Plot the curve $y = (x + 1)^3$ from $x = -5$ to $x = 2$. Draw the straight line from the origin tangential to the curve and determine its equation from the graph.

(9) Draw the graph of $y = x^3 - 2x^2 - 5x + 6$ from $x = -4$ to $x = 4$, and from it obtain the values of x for which $y = -2$.

(10) Draw the graph of $y = x^3 - 5x^2 + 7x - 3$ from $x = 0$ to $x = 4$. From your graph obtain the values of x for which $y = -\frac{1}{2}$.

(11) Draw the graphs of $x^2 + y^2 = 26$ and $2x + 3y + 13 = 0$ and use them to solve the equations simultaneously.

(12) Plot sufficient of the curves $x^2 + y^2 - 3x - 4 = 0$ and $x^2 = 3y$ to show all their points of intersection. Read the co-ordinates of x and y for the points of intersection, and deduce, from the original equations, equations for which these values of x and y are roots.

(13) Draw the graph of $x^2 + \frac{192}{x}$ from $x = 3$ to $x = 8$, and hence determine the least value.

(14) Plot the graphs of $y = 2^x$ and $y + 2 = 3^x$ from $x = -\frac{1}{2}$ to $x = 2$. Hence find the root of the equation $3^x - 2^x = 2$.

(15) Solve the following equation graphically—

$$4x^2 + 4x - 35 = 0.$$

Test your result by giving an algebraic solution. (*E.M.E.U.*)

(16) Draw the graph of $y = \frac{3}{x}$ for values of x between -3 and $+3$. With the same axes and scale draw the graph of $y = 2x + 1$ also between $x = -3$ and $x = +3$.

Find where the two graphs intersect and verify by calculation that the values of x where they intersect are the roots of the equation

$$2x + 1 = \frac{3}{x}. \qquad (N.C.)$$

(17) Plot to as large a scale as your graph paper will conveniently allow, the functions $3{\cdot}5 - \frac{1}{2}x^2$, and $\frac{2}{x}$ for values of x from -4 to $+4$, using the same scales and reference axes for both graphs. By means of these graphs, estimate to the nearest tenth each of the three values of x for which $x(3{\cdot}5 - \frac{1}{2}x^2) = 2$ (N.C.)

(18) The cost P(£) per mile of an electric cable is given by the expression

$$P = \frac{150}{x} + 650x$$

where x is the cross-sectional area of the cable in square inches. Taking values of x from 0·35 to 0·55 plot to as big a scale as is convenient P vertically against x horizontally.

From the graph find

(i) the cross-section for which the cost per mile is a minimum;
(ii) the minimum cost per mile. (U.E.I.)

(19) Solve graphically the following equation: $x^3 - 2x^2 - 2x + 1 = 0$. (U.E.I.)

(20) The total area A (in square feet) of the outside of a certain tank with a square base of side x ft. is given by the expression

$$A = x^2 + \frac{800}{x}.$$

Calculate A for values of x equal to 5, 6, 7, 8, 9 and 10. Plot A vertically and x horizontally. From the graph read off the value of x that will make A least. (U.E.I.)

(21) Show on a diagram how the expression $x^2 + \frac{30}{x}$ changes in value as x varies from 1 to 6.

From the diagram find—

(a) the least value of the expression,
(b) the value of x when the expression is least,
(c) the slope of the curve at $x = 4$. (U.E.I.)

(22) Graph the function $\frac{1}{3}(x-4)^2(x+2)$ for values of x from -2 to 6. By means of the graph solve each of the equations—

(i) $(x-4)^2(x+2) = 10$.
(ii) $(x-4)^2(x+2) = 17$. (N.C.)

(23) Solve graphically the equation $\varepsilon^x = 2x + 3$.
(Take values of x from -2 to $+2$.) (U.E.I.)

(24) Using squared paper find to three significant figures the values of x and t which satisfy the following equations—

$$3{\cdot}74\,\varepsilon^x = t^2$$
$$\cdot 5t + 2x = 5,$$

given that x lies between 1 and 2. (E.M.E.U.)

(25) The co-ordinates (x, y) of points on a certain curve are given by $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$, where ϕ is in radians. Calculate the co-ordinates of the points where $\phi = \frac{\pi}{6}$ and $\phi = \frac{3}{4}\pi$. (*U.L.C.I.*)

(26) Tabulate the values of r in the expression $r = a\cos\theta$ when $a = 2$, for values of θ from 0 to 2π radians. Hence, using polar co-ordinates, plot the curve $r = a\cos\theta$. Read off the values of θ in degrees when $r = 1{\cdot}5$. (*U.E.I.*)

(27) Plot in polar co-ordinates the curve $r = 2(1 - \cos\theta)$ from $\theta = -\pi$ to $\theta = \pi$ taking 1 in. as unit for r and angles at intervals of $\frac{\pi}{6}$. (*E.M.E.U.*)

(28) The profile of a cam forms part of the polar curve $r = 0{\cdot}287\,A$ where A is an angle in radians. Express r where A is in *degrees* and then plot the curve for values of A from $0°$ to $360°$.

From the diagram read off the value of A when $r = 1$. (*U.E.I.*)

CHAPTER VI

DIFFERENTIAL CALCULUS

SUPPOSE a car is moving along the road from London to Brighton at a steady rate of 20 m.p.h. and that at noon it is 5 miles from London. We know that the graph representing its distance y miles from London at time x hr. after noon will

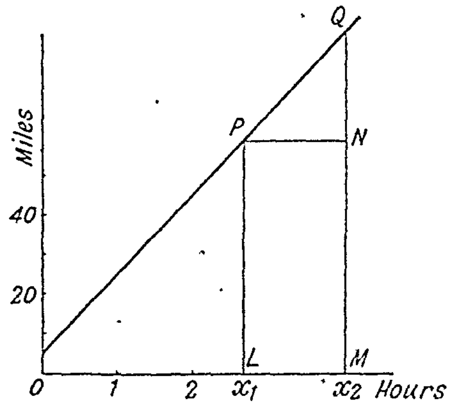


FIG. 78

be the straight line shown in Fig. 78, which can be represented by the equation

$$y = 20x + 5,$$

and that the slope of the line is 20 and gives the speed in miles per hour.

Further, taking any time interval L to M, the distances from London are given by LP and MQ, which are $20x_1 + 5$ and $20x_2 + 5$ where L is the time x_1 and M is the time x_2. The distance gone during this interval is the difference of these, namely NQ, or $20(x_2 - x_1)$, and the *average* speed for the time L to M will be obtained by dividing this distance by the time interval LM (or PN) which equals $(x_2 - x_1)$. Hence the average speed is 20 m.p.h. We have then, in the case of the straight-line graph, that the average rate of increase is constant and equal to the slope. This property does *not* hold for a curved graph, and we now proceed to extend the idea of a slope to such graphs.

The Slope of a Curve. Let P (Fig. 79) be any fixed point on the graph; we want to attach some meaning to the phrase "the slope of the curve at P." If Q is any other point on the graph the line through P and Q is called a *chord*. We will take sets of points Q_1, Q_2, . . . Q_1', Q_2', . . . and draw the chord through P and each of them in turn.

As the point Q is taken nearer and nearer to P we see that the chord approaches more and more closely to the position

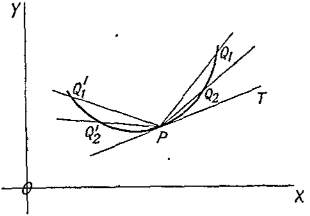


FIG. 79

PT, a line through P which grazes the curve there. This line PT is called the *tangent* at P to the curve, and the slope of PT is regarded as the *slope, or gradient, of the curve* at P. It is clear from this that the slope of a curve at a point on it will vary with the position of the point. To justify the idea of slope thus applied to a curve as suggesting a rate of increase, it will be clearer if we take a numerical example; to avoid hiding the argument under heavy arithmetic we will take one in which the numbers are very simple. We will take the curve as $y = x^2$ and the point P as having co-ordinates 1, 1.

Draw the lines parallel to the axes through P and Q (any other point on the curve) to form the triangle PQN, right-angled at N (Fig. 80).

In moving from P to Q on the curve, LM is the increase in the x co-ordinate and NQ is the increase in the y co-ordinate; the *average* increase for the interval L to M is $\frac{NQ}{LM}$ per unit x (or $\frac{NQ}{PN}$ since $LM = PN$). We will now make a table for various

intervals; we keep L at the point $x = 1$ and take intervals LM equal to ·1, ·01, etc., in turn. To help the calculation of y we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$ with a as 1 and b as the increase LM, so that

$$(1 + \cdot 1)^2 = 1{\cdot}21, \quad (1 + \cdot 01)^2 = 1{\cdot}0201, \text{ etc.}$$

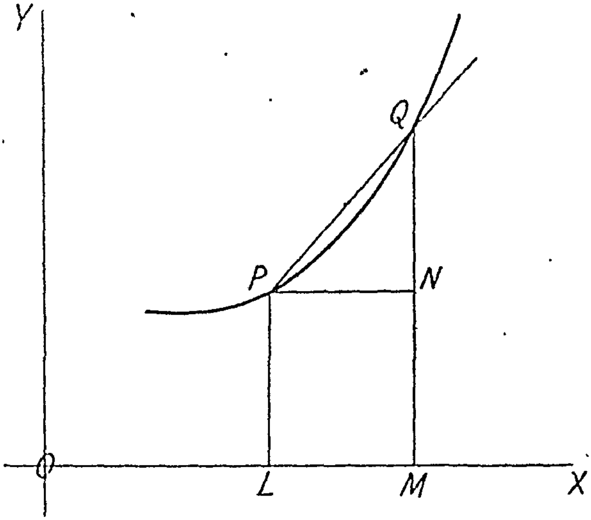


FIG. 80

Taking positions Q_1, Q_2, . . . of Q, which approach nearer and nearer to P, we can construct the table as follows—

	x co-ord. of Q	LM	y co-ord. of Q	NQ	average rate of increase $= \dfrac{NQ}{LM}$
Q_1	1 + ·1	·1	1·21	·21	2·1
Q_2	1 + ·01	·01	1·0201	·0201	2·01
Q_3	1 + ·001	·001	1·002001	·002001	2·001
....	....	....	....	....	....
....	....	....	....	....	....

It is clear that as we continue the process of bringing Q nearer and nearer to P the average rate of increase, as given in the last column, will approach more and more closely to 2—thus, for example, for the interval 1 to 1·00000001 the rate will be 2·00000001. We also notice that this average rate of increase, which has been obtained from the curve, is identical

with the slope of the chord. Hence we conclude that the rate of increase *at* P along the curve is identical with the slope of the tangent at P, and has (in this case) the numerical value 2. We speak of this rate of increase at P along the curve as the *slope of the curve at* P.

Note 1. The student will frequently find it necessary, in graphical work, to draw with a ruler the tangent at a point P on the curve. It is not always easy to do this with reasonable accuracy, and it is often helpful to mark on opposite sides of P two points Q, Q' equidistant from P but near to it (say ·1 in.); the tangent at P is roughly parallel to QQ'.

Note 2. The increase of y for a very small increase of x is very nearly (but not exactly) equal to the product of the gradient and the increase of x—thus if at a point $x = 4$ the gradient of a curve is 6, the increase of y when x is increased to 4·000002 is very nearly $6 \times {\cdot}000002 = {\cdot}000012$. This property is of the greatest importance in practical applications

Note 3. Although we have obtained the slope at a point on a curve from consideration of the average rate of increase over a very small interval, the student must not consider the slope at the point and this average rate as having exactly the same meaning In the case of a motor-car we can obtain an average rate in miles per hour from the distance gone during an interval of time, while the speedometer reading gives the true rate at any given time

Note 4. A decrease is, of course, a negative increase; thus, for example, a *decrease* of y of 4 per unit x is reckoned as an *increase* of -4 per unit x.

Later in the chapter we shall find rules by which the slopes of curves can be written down at sight. It is, however, advisable that the student should first work through some examples in detail in order that he should fully understand the principles involved; but it is not necessary to the process that he should construct a table of the type just used in order to find the slope of a curve. Thus—

EXAMPLE 1. Find the slope of the curve $2y = x^3$ at the point where $x = 4$.

When $x = 4$, $y = 32$; we take any point Q whose x co-ordinate is $4 + h$. If the y co-ordinate is $32 + k$ then

$$2\,(32 + k) = (4 + h)^3$$

where k, h are the lengths corresponding to NQ and LM in Fig. 80,

and $$64 + 2k = 64 + 48h + 12h^2 + h^3$$

using the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\therefore \qquad 2k = 48h + 12h^2 + h^3$$

or $$\frac{NQ}{LM} = \frac{k}{h} = 24 + 6h + \frac{h^2}{2}.$$

If we take Q very near to P, h is very small and $\frac{NQ}{LM}$ is very nearly equal to 24 (if h is very small indeed we can neglect $6h + \frac{h^2}{2}$ when compared with 24), or the slope of the curve $2y = x^3$ at the point where $x = 4$ is 24.

The term $6h + \frac{h^2}{2}$ corresponds to the difference between the average rate of increase over the interval 4 to $4 + h$ and the slope of the curve at $x = 4$ (see Note 3, page 156).

Example 2. For what value of x is the slope of $3y = x^2 + x$ equal to 5?

Let the co-ordinates of the desired point P (Fig. 80) be x_1, y_1 and the co-ordinates of Q be $x_1 + h$, $y_1 + k$.

Then, since P and Q are on the curve,

$$3y_1 = x_1^2 + x_1 \tag{1}$$

and

$$3(y_1 + k) = (x_1 + h)^2 + (x_1 + h)$$

or

$$3y_1 + 3k = x_1^2 + 2x_1h + h^2 + x_1 + h. \tag{2}$$

Subtracting (1) from (2) $3k = 2x_1h + h + h^2$

or

$$\frac{k}{h} = \frac{2x_1 + 1}{3} + \frac{h}{3}.$$

The slope at P is therefore $\frac{2x_1 + 1}{3}$ which equals 5 when $x_1 = 7$, or the x co-ordinate of the required point is 7.

EXAMPLES XXXIV

In Questions 1 to 4 find the slopes of the curves for the values of x indicated.

(1) $3y = x^2$ when $x = 2$.

(2) $y = 4x^3$ when $x = 1$.

(3) $y = 2x^2 + x$ when $x = 3$.

(4) $2y = x^2 - 1$ when $x = -1$.

(5) Find the co-ordinates of the point on $y = 3x^2 - 2x$ at which the slope is 1.

(6) Find the co-ordinates of the points on $2y = x^3$ at which y is increasing at the rate of 6 per unit x.

(7) Find the co-ordinates of the points on $y = 2x - x^3$ at which y is decreasing at the rate of 4 per unit x.

The Differential Calculus. The objects of the Differential Calculus are to provide rules by which the slopes of curves can be easily written down from their equations, and to use them to obtain properties of the curves which are of practical importance. For example, if we know that a certain equation

connects the efficiency of a motor with the number of revolutions per minute it is a matter of great importance that we should have some method of finding the speed which gives the maximum efficiency; or again, in the case of a loaded girder, that we should be able to calculate the increase of bending due to an increase of load. Problems of this type are dealt with most conveniently by means of the Differential Calculus.

Notation. In Chapter VII of Book I we introduced to the student the terms dependent and independent variables. In the bookwork we shall usually denote the independent variable by x and the dependent variable by y; y is then said to be a *function* of x and we regard the equation between them as giving us the value of y in terms of x. The relation may be expressed directly as in the case of $y = x^3$, or may be indirect as in the case of $x^2 + y^2 = 5$ It sometimes happens that we are not concerned with the exact form of the equation, and we then may write $y = f(x)$ (in words, "y equals a function of x") or $f(x, y) = 0$ (in words, "a function of x and y equals zero") respectively, to indicate that there is a relation between x and y.†

In the introductory paragraphs we have calculated slopes for definite numerical values of x (e.g. $x = 4$), but such calculations are too limited, and we now wish to find the slope for *any* value of x. It must, however, be clearly understood that, although we express the value of the co-ordinates by x, y, the x and y are definitely those of the point we are concerned with and have therefore definite values—they are no longer the general x and y of the co-ordinate system. In the previous work we denoted a small increase of x (the length $LM = PN$ in Fig. 80) by h; it will now be more convenient to use a symbol which clearly indicates an x increase, and we therefore express a small increase of x by δx (in words "delta x"). The increase of x gives rise to a small increase of y (the length NQ, previously denoted by k), which we express similarly by δy (in fact we use the symbol δ to express any increase; thus δx^3 implies an increase of x^3, δy^2 of y^2, and so on). The slope, or rate of increase of y with respect to x, at the point P (x, y) is denoted by the symbol $\frac{dy}{dx}$. This is also called the *differential coefficient*

† In this chapter we shall limit ourselves to the type $y = f(x)$ where the function is the sum of integral powers of x or is one of the simpler trigonometrical ratios. Harder cases will be dealt with in Book III.

of y with respect to x (or more briefly the *differential*) and is referred to in words as "*d y* by *d x*." *The student must fully realize that these are symbols to which algebraic manipulation can not be applied, thus δx is* NOT *δ times x, and $\frac{dy}{dx}$ is* NOT *a fraction and can* NOT *be simplified.*

Note. It is sometimes necessary to indicate that the slope is calculated for some particular point x_1, y_1 instead of for any point x, y on the curve. This can be represented conveniently by $\left(\frac{dy}{dx}\right)_1$, i.e. we have worked out the slope, $\frac{dy}{dx}$, for *any* point x, y and then replaced x and y by the particular co-ordinates x_1 and y_1.

The Differential of ax^n where n is a Positive Integer and a is any Number. We will first work out the value of $(x+h)^n$. We shall obtain it by considering repeated multiplications by the contracted method, regarding h as small, as set out in the working. It is clear that every time we multiply we increase the power of x by 1 and also increase the coefficient of the term containing h by 1, so that approximately

$$\begin{array}{rr|l}
 & x+h & \\
 & x+h & \\
\hline
 & x^2+xh & \\
 & xh & +h^2 \\
\hline
(x+h)^2 = & x^2+2xh & +h^2 \\
 & x+h & \\
\hline
 & x^3+2x^2h & +\text{ terms in } h^2 \\
 & x^2h & +\text{ ,, ,, ,, etc.} \\
\hline
(x+h)^3 = & x^3+3x^2h & +\text{ ,, ,, ,, ,,} \\
 & x+h & +\text{ ,, ,, ,, ,,} \\
\hline
 & x^4+3x^3h & +\text{ ,, ,, ,, ,,} \\
 & x^3h & +\text{ ,, ,, ,, ,,} \\
\hline
(x+h)^4 = & x^4+4x^3h & +\text{ ,, ,, ,, ,,} \\
 & x+h & +\text{ ,, ,, ,, ,,} \\
\hline
 & x^4+4x^4h & +\text{ ,, ,, ,, ,,} \\
 & x^4h & +\text{ ,, ,, ,, ,,} \\
\hline
(x+h)^5 = & x^5+5x^4h & +\text{ ,, ,, ,, ,,}
\end{array}$$

$$(x+h)^4 = x^4 + 4x^3h$$
$$(x+h)^{20} = x^{20} + 20x^{19}h$$

and

$$(x+h)^n = x^n + nx^{n-1}h;$$

or, accurately,

$$(x+h)^n = x^n + nx^{n-1}h + h^2R,$$

where h^2R represents the part of the multiplication which we have not worked out (R is some expression in x and h, starting off with the $(n-2)$th power of x and ending with h^{n-2}).

To return to our question: we have for the point P

$$y = ax^n,$$

and for a near point Q

$$y + \delta y = a\,(x + \delta x)^n,$$

or $$= a\,(x^n + nx^{n-1}\,\delta x + (\delta x)^2\,R).$$

Subtracting, $$\delta y = anx^{n-1}\,\delta x + a(\delta x)^2\,R,$$

or $$\frac{NQ}{LM} = \frac{\delta y}{\delta x} = anx^{n-1} + a(\delta x)\,R. \quad \text{(See Fig. 81.)}$$

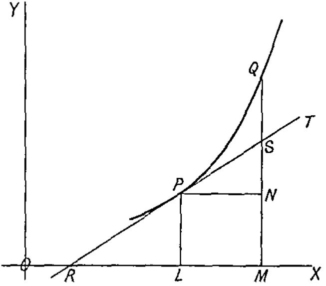


Fig. 81

As δx is made smaller and smaller (i.e. as Q is taken nearer and nearer to P) $\frac{NQ}{LM}$ approaches more and more closely to anx^{n-1} and we have therefore for the slope, or differential coefficient, of P

$$\frac{dy}{dx} = anx^{n-1}.$$

We can express the rule in words thus: *To differentiate a power of x, multiply the coefficient by the power and also reduce the power of x by* 1. We shall see later that the rule also holds when n is not a positive integer.

y is the Sum of Powers of x. If we have an expression y equal to the sum of powers of x, as, for example,

$$y = 3x^8 + 5x^4 + 2x^3 + 4x + 9,$$

it is clear that when x is given a small increase the resulting increase in y is equal to the sum of the increments of each of the terms, and therefore $\frac{dy}{dx}$ is equal to the sum of the differentials of the separate terms; thus in the example given

$$\begin{aligned}\frac{dy}{dx} &= 3 \,.\, 8x^7 + 5 \,.\, 4x^3 + 2 \,.\, 3x^2 + 4 \\ &= 24x^7 + 20x^3 + 6x^2 + 4.\end{aligned}$$

It is to be noted that the term 9, with no x in it, disappears on differentiation—a change in the value of x makes no change in the value of this term, or written symbolically

$$\frac{d9}{dx} = 0.$$

The differential of $4x$ is, of course, 4.

Fundamental Properties of $\frac{dy}{dx}$. Although our algebra has been applied to a particular case ($y = ax^n$) the properties of $\frac{dy}{dx}$ which may be deduced from Fig. 81 are quite general.

1. $\frac{dy}{dx}$ is the slope of the line PT and $= \frac{NS}{PN} = \frac{LP}{RL}$.

If the units of x and y are equal, as has been assumed without mention in the previous work to avoid confusion, $\frac{NS}{PN} =$ tan $N\hat{P}S$, or

$$\frac{dy}{dx} = \tan N\hat{P}S = \tan L\hat{R}P.$$

In practice we may wish to estimate the rate of increase of y by drawing the tangent to a graph in which the units are not equal, and measuring the angle. We must remember that NS and LP are measured in y units and PN and RL in x units. For example, if unit x is 3 in. and unit y is 4 in. and the angle LRP is 60°; then corresponding to a 3-in. increase in x (equal to 1 unit x) there is for the tangent RP, *on the graph*, an increase

of y equal to 3 tan 60°, which in terms of y units equals $\frac{3 \tan 60°}{4}$. Thus the slope of the tangent, and therefore of the graph at the point of contact, is $\frac{3 \tan 60°}{4}$.

$\therefore \frac{dy}{dx} = \frac{3 \tan 60°}{4}$ at the particular point.

2. For a small increase h of x equal to PN, the true increase in y is NQ and $h\frac{dy}{dx}$ is NS. The *error* in taking $h\frac{dy}{dx}$ as the increase of y for the small increase h of x is SQ. When Q is near to P this is negligible when compared with NQ, as may be seen by taking for Q on the figure a point on the curve, say, ·1 of an inch from P. We have, therefore, the important practical rule (as already stated in Note 2, page 156)—

For a small increase of x the increase of y is given approximately by the product of the differential coefficient and the small increase of x.

Note. The error corresponds to the term h^2R or $(\delta x)^2R$ in the algebra of page 159.

3. If $\frac{dy}{dx}$ is positive, the angle LRP is acute (less than 90°) and y increases as x increases;

If $\frac{dy}{dx}$ is negative, the angle LRP is obtuse (between 90° and 180°) and y decreases as x increases;

If $\frac{dy}{dx} = 0$, the angle LRP is zero and the tangent PT is parallel to the axis of x. The value of y at such a point is said to be *stationary*, as its change for a small change of x is negligible.

EXAMPLE 1. Differentiate $y = x^3 - 2x^2 + x + 6$. For what values of x is y (i) increasing (ii) decreasing (iii) stationary, as x increases?

Using the rule for differentiation of a power

$$y = x^3 - 2x^2 + x + 6$$

$$\frac{dy}{dx} = 3x^2 - 4x + 1.$$

Factorizing, we have $\frac{dy}{dx} = (3x - 1)(x - 1)$

and $$\frac{dy}{dx} = 0 \text{ when } x = \tfrac{1}{3} \text{ or } x = 1,$$

or the value of y is stationary when $x = \frac{1}{3}$ and when $x = 1$ (see Fig. 82).

If x is less than $\frac{1}{3}$, both $3x - 1$ and $x - 1$ are negative, and the product is therefore positive.

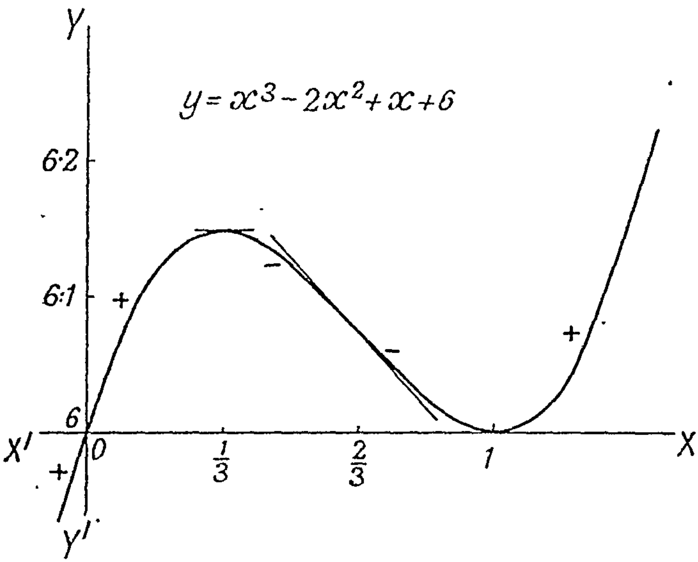


FIG. 82

If x is greater than 1, both $3x - 1$ and $x - 1$ are positive and the product is therefore positive.

Hence, for x less than $\frac{1}{3}$ and for x greater than 1, y increases as x increases.

If x is greater than $\frac{1}{3}$ but less than 1, $3x - 1$ is positive and $x - 1$ is negative, and the product is therefore negative; hence, for x between $\frac{1}{3}$ and 1, y decreases as x increases.

The sign of $\frac{dy}{dx}$ for various parts of the curve is shown in the figure. It will be seen that where $\frac{dy}{dx}$ is positive the angle between the tangent and the positive direction of the x axis is

acute; where $\frac{dy}{dx}$ is negative the angle is obtuse; and where $\frac{dy}{dx}$ $= 0$ (at $x = \frac{1}{3}$ or 1) the tangent is parallel to the x axis.

EXAMPLE 2. The height h ft. of a ball thrown upwards is given at time t sec. by the equation $h = 64t - 16t^2$. Find the velocity and acceleration at time t.

Now the velocity v is the rate of increase of height,

or $$v = \frac{dh}{dt}.$$

But $$h = 64t - 16t^2$$

$$\therefore \quad \frac{dh}{dt} = 64 - 32t$$

or, for the velocity, $v = 64 - 32t$ in feet per second. When $t = 2$, $v = 0$ and the ball is instantaneously at rest and is at its greatest height.

Again, the acceleration f is the rate of increase of velocity,

or $$f = \frac{dv}{dt}.$$

But $$v = 64 - 32t$$

$$\therefore \quad \frac{dv}{dt} = -32$$

or, for the acceleration, $f = -32$ in feet per second per second. Since f is negative v *decreases* as t increases; the ball is said to have a *retardation* of 32 ft. per second per second.

EXAMPLE 3. The radius of a sphere is taken to be 2 ft. and the volume is calculated. If the radius is actually $\frac{1}{100}$ in. less than this, by how much per cent is the calculated volume too big?

If V cub. in. is the volume of a sphere of radius r in.

$$V = \frac{4}{3}\pi r^3$$

and $$\frac{dV}{dr} = \frac{4}{3}\pi \, . \, 3r^2 = 4\pi r^2.$$

Hence, if the radius is increased by a small amount δr, the volume is increased by a small amount δV where

$$\delta V = 4\pi r^2 \delta r.$$

In the example $\delta r = -\frac{1}{100}$ and $r = 24$ inches.

$$\therefore \qquad \delta V = -\frac{4\pi(24)^2}{100}.$$

The calculated value is therefore too small by $-\frac{4\pi(24)^2}{100}$ cub. in., i.e. $+\frac{4\pi(24)^2}{100}$ too big,

or
$$\frac{100 \times \frac{4\pi(24)^2}{100}}{\frac{4}{3}\pi(24)^3} = \frac{3}{24} = \frac{1}{8} \text{ per cent.}$$

EXAMPLE 4. Figure 83 shows a vessel with plane sloping sides and 3 in. deep. The bottom is a rectangle 20 in. by 15 in.

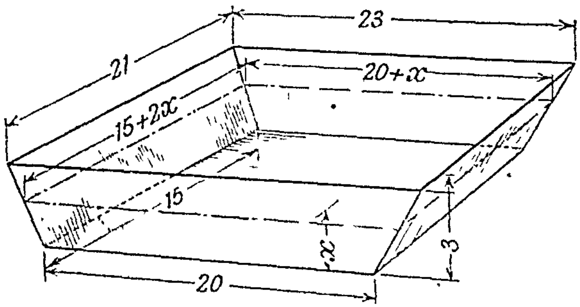


FIG. 83

and the top is a parallel rectangle 23 in. by 21 in. Water is poured in at the rate of 22 cub. in. per minute. Find the rate at which the depth of the water is increasing when it is 2 in.

Suppose the depth of the water is x in.; the surface is a rectangle whose length is $20 + x$ in. (the length increases from 20 in. to 23 in. in 3-in. depth, i.e. by 1 in. per inch depth) and whose breadth is $15 + 2x$ (the breadth increases from 15 in. to 21 in. in 3-in. depth, i.e. by 2 in. per inch depth), or the area A is given by

$$A = (20 + x)(15 + 2x) = (300 + 55x + 2x^2) \text{ sq. in.}$$

Suppose that in a very short time δt the depth increases by δx; the increase in volume is then very nearly the volume of a slice of area A and thickness δx, and is given by

$$A\,\delta x = (300 + 55x + 2x^2)\,\delta x \text{ cub. in. approximately.}$$

But 22 cub. in. are poured in per minute and therefore the increase in volume in δt min. is $22\,\delta t$ cub. in. We have then, very nearly,

$$(300 + 55x + 2x^2)\,\delta x = 22\,\delta t$$

or

$$\frac{dx}{dt} = \frac{22}{300 + 55x + 2x^2}.$$

When $x = 2$ the rate $\left(\frac{dx}{dt}\right)$ at which the depth is increasing is

$$\frac{22}{300 + 55\,.\,2 + 2\,.\,2^2} = \frac{22}{418} = \frac{1}{19}\text{ in. per minute.}$$

EXAMPLES XXXV

In Questions 1 to 16 find the differentials of the given expressions.

(1) x^7. (2) $3x^4$. (3) $5x^5$. (4) $7x^{14}$.
(5) $-3x^3$. (6) $2x + 3x^2$. (7) $4x - 5$. (8) $x^3 - 2x - 7$.
(9) $x^4 - 3x^2 + 5$ (10) $8x^7 - 7x^8$ (11) $x^{12} - 12x$ (12) $(1 + 2x)^2$.
(13) $(x^2 + x)^4$. (14) $(1 - x)^3$.
(15) $(2 - x^2)^3$. (16) $x^3\left(1 - \frac{1}{x}\right)^2$

(17) A point P moves along a straight line AB, the distance AP being x ft. at time t sec If $x = 7 + 18t - t^3$, find the velocity and acceleration of P (i) when $t = 2$, (ii) when $t = 4$

(18) A road runs east and west, the distance s yd. east of a fixed point O on the road at time t min. of a man on the road is given by the equation $s = 18t - 3t^2$ for times t from 0 to 20. From $t = 20$ he walks with constant speed. Find

(i) the velocity when $t = 1$,
(ii) the time when the man starts walking west,
(iii) the distance he goes east,
(iv) the velocity when $t = 20$,
(v) the equation giving his position for t greater than 20.

(19) The radius of a circle is 7 in. Due to a temperature change, the radius increases by 0·0003 in. By means of the calculus calculate the increase in area. (Take π to be $\frac{22}{7}$.)

(20) The radius of a sphere is increasing at the rate of 0·014 in. per sec. Find the rate of increase of the area when the radius is 8 in. (Take π to be $\frac{22}{7}$.)

(21) At a point P on the curve $y = 3x^3 - 2x$, $x = 4$. Find approximately the y co-ordinate of the point whose x co-ordinate exceeds that of P by 0·02.

(22) At the point P on the curve $2y = x^4 - x^3$, $x = 2$. Find approximately by how much the x co-ordinate of P must be increased to increase the y co-ordinate by 0·28.

(23) The sides of an equilateral triangle are increasing at the rate of b cm. per sec. At what rate is the area increasing when the side is a cm.

(24) An inverted hollow cone has height h in. and base radius a in. If water is being poured in at the rate of V cub. in. per second at what rate is the surface rising when the depth is x in.?

The Tangent to a Curve. Suppose we want the equation of the tangent to a curve, whose equation is known, at a point on it whose co-ordinates are x_1, y_1. We first work out the slope $\frac{dy}{dx}$ for *any* point x, y, and by substitution obtain the slope $\left(\frac{dy}{dx}\right)_1$ at the *particular point* x_1, y_1. The equation of the tangent is then

$$(y - y_1) = \left(\frac{dy}{dx}\right)_1 (x - x_1)$$

because

(i) this equation is of the first degree in x and y and therefore represents a straight line;

(ii) the straight line goes through the point x_1, y_1 since, if we put x_1 for x and y_1 for y, both sides are zero;

(iii) the slope of the line is $\left(\frac{dy}{dx}\right)_1$, which is the same as that of the curve at x_1, y_1.

EXAMPLE 1. Find the equation of the tangent to the curve at the point where $x = 3$ on $y = 2x^2 - 7x + 1$.

When $x = 3$, $y = -2$.

$$\frac{dy}{dx} = 4x - 7. \qquad \therefore \left(\frac{dy}{dx}\right)_1 = 4 \,.\, 3 - 7 = 5.$$

The equation of the tangent is therefore

$$(y + 2) = 5(x - 3),$$

or

$$y = 5x - 17.$$

EXAMPLE 2. Find the co-ordinates of the points on the curve $y = x^2 + 2x + 4$ at which the tangents pass through the origin.

$$\frac{dy}{dx} = 2x + 2$$

$\therefore$ the equation of the tangent at x_1, y_1 is

$$(y - y_1) = (2x_1 + 2)(x - x_1)$$

which passes through the origin (0, 0) if

$$y_1 = (2x_1 + 2)\,x_1.$$

Solving with $y_1 = x_1^2 + 2x_1 + 4$ (since x_1, y_1 lies on the curve) we have either $x_1 = 2$, $y_1 = 12$, or $x_1 = -2$, $y_1 = 4$.

Points of Inflexion. Suppose we take the curve $y = x^3 - 2x^2 + x + 6$ given in Example 1, page 162. The slope at any point is given by

$$\frac{dy}{dx} = 3x^2 - 4x + 1.$$

We have already found that the slope is zero when $x = \frac{1}{3}$ and $x = 1$. It is in itself a function of x and we can therefore discover how *it* varies with x (increasing, etc.) by considering *its* differential, which is $6x - 4$.

$6x - 4 = 0$ when $x = \frac{2}{3}$, or the slope is stationary at $x = \frac{2}{3}$.

When x is less than $\frac{2}{3}$, $6x - 4$ is negative, or the slope is decreasing.

When x is greater than $\frac{2}{3}$, $6x - 4$ is positive, or the slope is increasing.

Referring to Fig. 82, and starting from the left, we see that the slope decreases as x increases to $\frac{2}{3}$ (at $x = \frac{1}{3}$ it is zero and from $\frac{1}{3}$ to $\frac{2}{3}$ it is actually negative); from $x = \frac{2}{3}$ the slope increases from the negative value at $\frac{2}{3}$, becoming zero again at $x = 1$, after which it is positive.

(a) (b) (c)

FIG. 84

A point where the slope is stationary is called a *point of inflexion.* Three sketches of inflexions are given in Fig. 84, and it will be seen that the direction of bending of the curve changes at the inflexion and that the curve actually crosses the tangent there. If we regard the curves in the figure as giving sections of a roadway we see that the inflexion gives

in case (*a*) the point of greatest steepness,

„ (*b*) the point of least steepness,

„ (*c*) the point where the road is flat (the slope eases off to this point and then steepens).

From $x = \frac{1}{3}$ to $x = 1$ we can regard the example worked out in detail as a case of (a), the negative slope implying descent as x increases.

The differential of the differential $\frac{dy}{dx}$ is usually denoted by the special symbol $\frac{d^2y}{dx^2}$ (referred to in words as "d squared y by $d\,x$ squared," though no squares are involved) and we have the rule

To find the points of inflexion find the points where $\frac{d^2y}{dx^2} = 0$.

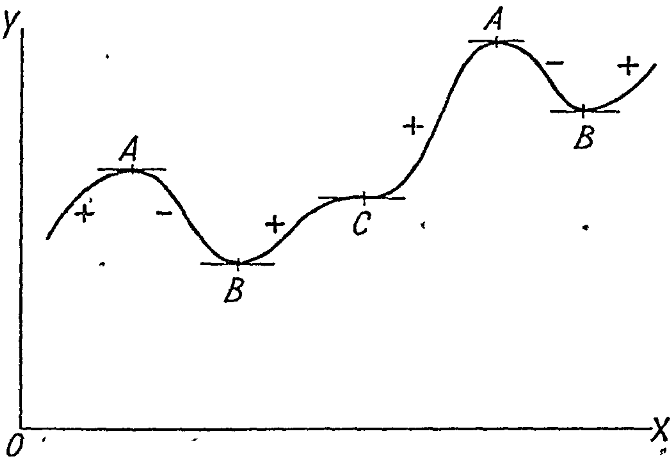


FIG. 85

Points of inflexion have considerable importance in engineering design; for example, a vertical supporting pillar becomes slightly bent under load and will be unsafe if the curve which it assumes contains a point of inflexion.

Maxima and Minima. We have already defined a maximum value of y (Chapter V, page 147) as one which is greater than that for any of the nearby values of x and a minimum as one which is less. In Fig. 85 the points A are maxima and the points B give minima; the point C $\left(\text{which is a point of inflexion, } \frac{d^2y}{dx^2} = 0\right)$ is neither a maximum nor a minimum, as y is greater at C than it is at a point just to the left and less than it is at a point just to the right. Maximum and minimum do not necessarily

mean absolute greatest and least—in the figure the second minimum is actually greater than the first maximum.

It is clear that at all these points the tangent is parallel to the x axis, or $\frac{dy}{dx} = 0$.

The sign of $\frac{dy}{dx}$ for various sections of the curve (+ or −) is marked on the figure; as we move to the right x increases and we have—

While passing through an A point $\frac{dy}{dx}$ *decreases* from + to −,

or at a maximum $\frac{d^2y}{dx^2}$ is negative.

While passing through a B point $\frac{dy}{dx}$ *increases* from − to +,

or at a minimum $\frac{d^2y}{dx^2}$ is positive.

At the inflexional point C, $\frac{dy}{dx}$ is zero but there is no change of sign in passing through it; $\frac{d^2y}{dx^2}$ is zero there.

We have therefore the rule—

To find the positions which give maxima and minima first find where $\frac{dy}{dx} = 0$*; if the co-ordinates of such a point make* $\frac{d^2y}{dx^2}$ *negative we have a maximum; if they make it positive we have a minimum.*

Example 1. Find the maximum and minimum values (if any) of $x^3 - 2x^2 + x + 6$.

Writing y for the expression, $\frac{dy}{dx} = 3x^2 - 4x + 1$;

$$3x^2 - 4x + 1 = 0 \text{ when } x = \tfrac{1}{3} \text{ and } x = 1;$$

$$\frac{d^2y}{dx^2} = 6x - 4.$$

When $x = \frac{1}{3}$ $\quad \frac{d^2y}{dx^2} = -2$, which is negative.

When $x = 1 \quad \dfrac{d^2y}{dx^2} = 2$, which is positive.

Hence $x = \frac{1}{3}$ gives the maximum value for $y, = 6\frac{4}{27}$,
and $x = 1$ gives the minimum value for $y, = 6$.

(Compare Fig. 82, page 163; it will be seen that the maximum and minimum are not greatest and least values.)

Example 2. Find the maximum and minimum values (if any) of $x^3 + 2x^2 + 6x - 1$.

Writing y for the expression, $\dfrac{dy}{dx} = 3x^2 + 4x + 6$; the equation $3x^2 + 4x + 6 = 0$ has no real roots, and there is therefore no maximum or minimum value of y.

Example 3. Find the maximum and minimum values (if any) of $x^4 - 5x^3 + 9x^2 - 7x + 2$.

Writing y for the expression, $\dfrac{dy}{dx} = 4x^3 - 15x^2 + 18x - 7$; by trial $x - 1$ is a factor of $4x^3 - 15x^2 + 18x - 7$, which equals $(x - 1)(4x^2 - 11x + 7) = (x - 1)^2 (4x - 7)$, or $\dfrac{dy}{dx} = 0$ when $x = 1$ and when $x = 1\frac{3}{4}$.

$$\frac{d^2y}{dx^2} = 12x^2 - 30x + 18.$$

When $x = 1, \dfrac{d^2y}{dx^2} = 0$.

When $x = 1\frac{3}{4}, \dfrac{d^2y}{dx^2} = 2\frac{1}{4}$, which is positive.

Hence $x = 1\frac{3}{4}$ gives the minimum value for $y, = \dfrac{27}{256}$.

There is no other maximum or minimum, $x = 1$ giving a stationary value at an inflexion.

(Harder examples on maxima and minima will be given in Book III.)

EXAMPLES XXXVI

(1) Find the equation of the tangent at the point on $y = 2x^2$ where $x = 2$.

(2) Find the equation of the tangent at the point on $y = x^4 + 2x^2$ where $x = 1$.

(3) Find the equation of the tangent at the point on $y = x^3 - 2x$ where $x = 2$.

(4) Find the equations of the tangents to $y = 3x^2$ which are parallel to the line $y = x$.

(5) Find the co-ordinates of the point of inflexion on $y = x^3 - 3x^2 + x - 7$.

(6) Find the x co-ordinates of the points of inflexion on the curve $y = x^4 - 4x^3 - 18x^2 - 3$.

(7) Find the values of x which give maximum and minimum values to $x^3 - 3x^2 - 9x$; distinguish between them

(8) Find the maximum and minimum values of $4 + x - x^2 - x^3$.

(9) Find the maximum and minimum values of $x^4 - 2x^3 + 2x + 1$.

(10) Find the values of x for which the tangent to $y = x^4 - 6x^2 + 8x$ is parallel to the x axis, and determine whether they correspond to points of inflexion or maximum or minimum values of y

The Differentials of sin $(ax + b)$ and cos $(ax + b)$ where a and b are any Numbers. In applications of the calculus to

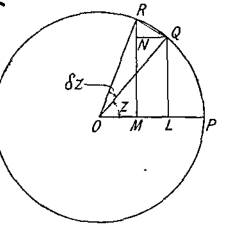


FIG. 86

trigonometric ratios it is *essential* that all angles should be measured in *radians*.

Let $ax + b = z$, then, for a small increase δx of x, we shall have a small increase δz of z where $a\delta x = \delta z$.

Construct the Fig. 86 in which the circle (centre O) has radius 1, $P\hat{O}Q = z$, and $Q\hat{O}R = \delta z$. (For the sake of clearness in the figure the angle δz has not actually been drawn very small); QL, RM are perpendicular to OP, and QN is perpendicular to RM.

Since δz is measured in radians and the radius is 1

$$\text{the arc } RQ = 1 \,.\, \delta z = \delta z$$

It is this step which makes the use of radian measure essential.

Since δz is small

the chord RQ very nearly equals the arc $RQ = \delta z$.

In the isosceles triangle ORQ,

$$O\hat{R}Q = O\hat{Q}R = \tfrac{1}{2}(\pi - R\hat{O}Q) = \frac{\pi}{2} - \tfrac{1}{2}\delta z.$$

In the right-angled triangle ORM,

$$O\hat{R}M = \frac{\pi}{2} - R\hat{O}M = \frac{\pi}{2} - z - \delta z;$$

$$\therefore \quad N\hat{R}Q = O\hat{R}Q - O\hat{R}M = z + \tfrac{1}{2}\delta z.$$

We use π radians instead of 180° as all angles are measured in radians.

Since the radius is 1, $\sin(z + \delta z) = RM$, and $\sin z = QL$.

$\therefore RN = RM - NM = RM - QL = \sin(z + \delta z) - \sin z$,

but $\quad \delta(\sin z)$ is $\sin(z + \delta z) - \sin z$;

$\therefore \quad \delta(\sin z) = RN$.

Now $\quad \sin(ax + b)$ is $\sin z$ and $a\delta x = \delta z$;

$$\therefore \quad \frac{\delta[\sin(ax + b)]}{\delta x} = \frac{a\,\delta(\sin z)}{\delta z} = \frac{aRN}{RQ}$$

approximately, where RQ is the chord,

$$= a\cos Q\hat{R}N = a\cos\left(z + \frac{\delta z}{2}\right),$$

or, since $z + \dfrac{\delta z}{2}$, when δz is very small, nearly equals z, i.e. $ax + b$,

$$\boldsymbol{\frac{d[\sin(ax + b)]}{dx} = a\cos(ax + b).}$$

Again, $\quad \cos(z + \delta z) = OM$ and $\cos z = OL$;

$\therefore \quad NQ = ML = OL - OM = \cos z - \cos(z + \delta z)$,

but $\quad \delta(\cos z)$ is $\cos(z + \delta z) - \cos z$:

$\therefore \quad \delta(\cos z)$ is $-NQ$

and, as before,

$$\frac{\delta[\cos(ax + b)]}{\delta x} = \frac{a\delta(\cos z)}{\delta z} = -\frac{aNQ}{RQ} \text{ approximately,}$$

$$= -a\sin Q\hat{R}N = -a\sin\left(z + \frac{\delta z}{2}\right).$$

Hence, finally,

$$\frac{d\,[\cos(ax+b)]}{dx} = -a\sin(ax+b).$$

EXAMPLE 1. If $y = \sin 4x$ and x is measured in degrees find $\frac{dy}{dx}$ when $x = 30°$.

Before we can differentiate we must turn the angle into radians; $1° = \frac{\pi}{180}$ radians, so, in radians,

$$y = \sin\frac{4\pi}{180}x \text{ radians}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{4\pi}{180}\cos\frac{4\pi}{180}x \text{ radians} \qquad \text{(from the formula for sine)}$$

$$= \frac{4\pi}{180}\cos 4x°.$$

When $x = 30°$, $\cos 4x° = \cos 120° = -\frac{1}{2}$,

and $$\frac{dy}{dx} = -\frac{4\pi}{180 \times 2} = -\frac{\pi}{90}.$$

EXAMPLE 2. For what values of x less than 2 radians is the tangent to the graph of $y = 2\sin x + \cos 2x$ parallel to the axis of x?

$$y = 2\sin x + \cos 2x$$

$$\therefore \qquad \frac{dy}{dx} = 2\cos x - 2\sin 2x \qquad \text{(from the two formulae)}$$

But $\sin 2x = 2\sin x\cos x \quad \therefore \frac{dy}{dx} = 2\cos x\,(1 - 2\sin x)$

or $$\frac{dy}{dx} = 0 \text{ when } \cos x = 0, \text{ i.e. } x = \frac{\pi}{2},$$

$$\text{or} \qquad \sin x = \tfrac{1}{2}, \text{ i.e. } x = \frac{\pi}{6}.$$

These are the only values of x less than 2 which satisfy the equations.

Again $\frac{d^2y}{dx^2} = -2\sin x - 4\cos 2x$

when $x = \frac{\pi}{2}$, $\frac{d^2y}{dx^2} = -2 \cdot 1 - 4(-1) = 2$, which is positive and the value of y is a minimum;

When $x = \frac{\pi}{6}$, $\frac{d^2y}{dx^2} = -2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2}$, which is negative and the value of y is a maximum.

EXAMPLES XXXVII

(The angles are in radians unless otherwise shown.)

In Questions 1 to 11 obtain the differential coefficients of

(1) $\sin(3x + 4)$. (2) $\cos 5x$. (3) $3 \sin(4 - 2x)$.

(4) $\cos(5 - 7x)$. (5) $\cos 3x°$. (6) $4 \sin\left(3x - \frac{\pi}{4}\right)$.

(7) $2 \cos(3x - 7)$. (8) $\sin(6x - 20)°$. (9) $2 \cos\left(2x - \frac{\pi}{6}\right)$.

(10) $\cos\left(\frac{\pi}{3} - 2x\right)$. (11) $\cos 2x + \sin 2x$.

(12) As x varies from 0 to $\pi/2$ radians, find the value of x which makes $\sin x + 2 \cos x$ a maximum.

(13) As x varies from 0 to $\pi/2$ radians, find the value of x which makes $2 \sin x + \cos x$ a maximum.

(14) As x varies from 0 to π radians, find the value of x which makes $\cos x - 3 \sin x$ a minimum.

(15) As x varies from 0 to π radians, find the value of x which makes $2 \sin x - \sin 2x$ a maximum.

(16) As x varies from 0 to $\pi/2$ radians, find the value of x which makes $\cos 3x - 3 \cos x$ a minimum.

(Additional elementary examples on the work of this chapter will be found on page 178.)

EXAMPLES XXXVIII

(1) Brass increases in length by ·000019 of its length for each degree centigrade increase in temperature. What is the volume of a cube of brass at 6° C. if at 1° C. each side measures 2 cm.?

(2) The distance moved s at time t by a body is given by $s = 7t - 8t^2 + 3t^3$. At what times is it at rest and at what time is the acceleration zero?

(3) An inverted hollow cone has height h in. and base-radius a in. Water is being poured into it and the depth is increasing at the uniform rate of b in. per sec. What is the rate at which the water is being poured in when the depth is x in.?

(4) Find the equation of the tangent to $4y = x^2$ which is parallel to the line $y = 3x$.

(5) Find the value of x for which the excess of $2x + 4$ over $4x - x^2$ is a minimum.

(6) Find the values of x which give the maximum and minimum values of $y = x^3 - 5x^2 + 8x$, and also that which gives the inflexion.

(7) Show that the graph of $y = x^3 - 3x^2 + 5x$ has no stationary point but that it has a point of inflexion.

(8) (*a*) The distance s (ft.) moved through by a body in time t (secs.) is given by the expression

$$s = 200t - 16t^2.$$

Find the velocity $\left(\frac{ds}{dt}\right)$ at $t = 0$ and $t = 4$ sec.

When will the body be at rest?

(*b*) Find the maximum value of the function $1 + 4x - 2x^2$. (*U.E.I.*)

(9) Prove that the turning values of the cubic function

$$a(x^3 - 9x^2 + 15x) + b,$$

where a and b are constants, occur when x has the values $+1$ and $+5$ respectively.

If this function has turning values of $+4$ and -2 when x has the values $+1$ and $+5$ respectively, find the values of a and b and calculate the values of the function when x equals -1, $+3$, $+7$. (*N.C.*)

(10) The values of the function $ax^2 + bx + c$ are 6·6 and 8·2 when x equals 2 and 2·8 respectively. The slope of the graph of the function when $x = 2·8$ is treble its slope when $x = 2$. Find the constants a, b and c. (*N.C.*)

(11) A curve of the form $y = ax + bx^2 + cx^3$ passes through the point (1, 1) and has turning values when $x = 2$ and $x = 3$. Find a, b and c.

(*E.M.E.U.*)

(12) An open tank with a square base is to be constructed to hold 108 cu. ft of water. Find its dimensions if a minimum amount of material is used.

(*E.M.E.U.*)

ADDITIONAL ELEMENTARY EXAMPLES

Chapter I

(1) Solve the equations—

(1) $\dfrac{x}{2x-1} + \dfrac{5}{x+3} = \dfrac{1}{2}$.

(2) $x - 3y + 7z = 12,$
$z - x + y = 0,$
$2y - 3x + 2z = 1.$

(2) Solve the equations (giving approximate answers to three significant figures)—

(1) $4x^2 + 8x - 17 = 0;$
(2) $0{\cdot}2x^2 + 1{\cdot}6x + 0{\cdot}3 = 0.$

(3) Solve the equations—

(1) $y - 4x = 0,\ 4x^2 + 5xy + 6y^2 = 270;$

(2) $ax - by = 0,\ ax^2 + by^2 = \dfrac{a+b}{ab}.$

(4) Solve the equations—

(1) $2^y - 3^x = 2,\ 2^{y+1} - 3^{x+1} = 0;$

(2) $x^2 + 2xy = 3;\ 3x + 2y = 5.$

(5) A man sells 10 tons more of a cheaper coal than he does of a dearer, which costs 12s. per ton more. He receives in all £83. If he had sold them both at the lower rate, he would have received only £74. How many tons of each sort does he sell and what are the costs per ton?

(6) A ball is thrown downwards with a starting velocity of 70 ft. per sec. The distance gone, x ft., after t sec. is given by $x = 70t + 16t^2$. Find the time taken for a distance of 200 ft.

(7) A rectangle is 2 in. longer than it is broad. If the length and breadth are both increased by 3 in., the area is increased by 69 sq. in. Find the original length.

Chapter II

(8) Prove that $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$.

(9) Prove that $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$.

(10) Prove that $\dfrac{\sin A}{\sin A - \cos A} + \dfrac{\sin A}{\sin A + \cos A} = 2 + \dfrac{2}{\sec^2 A - 2}$.

(11) Solve $3\sin^2\theta - \cos\theta = 1$.

(12) A man walks 3 miles in the direction N. 42° E. and then 2 miles in the direction S. 10° E. Calculate the distance and bearing of his final position from his starting point.

(13) Solve the triangle in which $\hat{B} = 28°$, $\hat{C} = 40°$, $a = 5$.

(14) Solve the triangle in which $\hat{A} = 30°$, $a = 4$, $b = 7$.

(15) If $\operatorname{cosec} A = \frac{13}{12}$, calculate without the use of tables the values of $\cos A$, $\sin 2A$, $\cos 2A$.

(16) Angles X and Y, both less than 90°, are such that $\sin X = \frac{3}{5}$,
$\tan Y = \frac{5}{12}$.
Without the use of tables, find the values of $\sin(X + Y)$ and
$\tan(X - Y)$.

(17) Express the following in terms of the trigonometrical ratios of the angle A—

$$\sin(270° - A),\ \cos(270° + A),\ \tan(180° + A).$$

(18) Prove that—

$$\cos(90° + A)\cos(180° - B) + \cos(180° + A)\cos(270° - B) = \sin(A + B).$$

Chapter III

(19) Forces act on a body of magnitudes and directions East of North as follows: 2 lb., 70°; 8 lb., 90°; 4 lb., 150°; 7 lb., 236°. Calculate the magnitude and direction of the resultant force.

(20) Check the previous question by means of the polygon of forces.

(21) Vectors of magnitudes 7, 8, 6, 4 and 5 units have directions N., N.W., S.E., W. and S.W. respectively. What is the magnitude and direction of the resultant vector when the last three vectors are subtracted from the sum of the first two?

(22) A body which is moving at 20 ft. per sec. due north is given an acceleration of 2 ft. per sec. per second in direction North West for 3 sec. and then an acceleration of 3 ft. per sec. per second in direction N. 58° W. for 3 sec. Find its final velocity.

(23) A man in a boat crosses a river 250 ft. wide, the velocity of the current being 6 m.p.h. He reaches the opposite bank 100 ft. downstream from his starting point. In what direction did he row if his velocity of rowing is 6 m.p.h.?

Chapter IV

(24) Ten pennies (each of diameter 1·2 in.) are arranged on a plane to touch each other with their centres lying in a circle. Calculate the radii of the circles (i) on which the points of contact lie, (ii) on which the centres lie.

(25) A cap is cut off a sphere by a plane. The height of the cap is 1 in. and the radius of its base is 3 in. Calculate the radius of the sphere, the area of the curved surface of the cap, and the volume of the cap.

(26) A tent is in the shape of a right circular cone. The area of its circular base is 100 sq. ft. and its volume is 350 cu. ft. Find to the nearest square foot the area of its curved surface.

(27) A right circular cone has radius 3 in. and height 4 in. Find the vertical angle, the volume, and the area of the curved surface. If a piece of paper exactly covering the curved surface is laid flat, what is the angle between the extreme radii?

(28) A crossed belt tightly surrounds two wheels in the same plane. The radii of the wheels are 5 and 8 in., and their centres are 17 in. apart. Calculate the length of the belt.

Chapter V

(29) Draw, on the same scale and axes, the graphs of $y = 1 + 4x - x^2$ and $xy - x - y - 3 = 0$ from $x = -1$ to $x = 4$. Write down the x co-ordinates of their points of intersection.

(30) Draw the graph of $y = x^3 - 2x^2 - 5x + 6$ from $x = -4$ to $x = +4$, and from the graph obtain the values of x for which $y = -1$.

(31) Plot the curves $y = x^3$, $2y + 3x = 6$ from $x = 0$ to $x = 2\frac{1}{2}$. Hence find a root of the equation $2x^3 + 3x - 6 = 0$ to one place of decimals.

(32) Plot the curve $y = (x + 1)^3$ from $x = -5$ to $x = 2$, and draw the straight line from the origin tangential to the curve. Determine its equation from the graph. From the two equations obtain an equation in x of which the x co-ordinates of the intersections of the line and the curve are the roots.

Chapter VI

(33) Find the x co-ordinates of the points on $y = x^3 - 2x^2 + 2x$ at which the gradient is six times the gradient at $x = 1$.

(34) The line $x = a$ cuts $y = x^2 + 3x$ and $y = x - x^2$ in points where the tangents to the curves are parallel. Find the value of a.

(35) Obtain the equation of the tangent to $y = 2x^2 - x^3$ at the point whose x co-ordinate is x_1. Hence obtain the equation of the line through the origin which touches the curve elsewhere.

(36) Find the values of x which give the maximum and minimum values of $y = x^3 - 3x^2 - 24x - 4$, and also that which gives the point of inflexion.

(37) Show that the graph of $y = 3x^3 - 3x^2 + 2x$ has no stationary point, but that it has a point of inflexion.

(38) Show that the graph of $y = 3x^3 - 3x^2 + x$ has no maximum or minimum, but that it has a point of inflexion.

(39) Obtain the differential coefficients of $\sin(4 - 2x)$, $\cos(4 - 2x)$, $\sin(120 + 5x)^\circ$.

MATHEMATICAL TABLES

AND

TABLE OF CONSTANTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	1	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	11	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	13
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	13
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993					
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998					
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000					

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	9999	9999	9999	9999					
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995					
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987					
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	1
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	4	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

The black type indicates that the integer changes.

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	13
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	13
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	11	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	·1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

7A—(T.12)

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	0·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	0·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	0·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	0·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	0·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	0·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	0·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	0·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	0·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	0·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	0·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	0·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	0·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	0·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	0·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	0·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	0·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	0·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	0·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	0·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	0·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	0·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	0·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	0·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	0·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	0·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	0·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	**0057**	**0145**	**0233**	**0323**	**0413**	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	91
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2·9042	9208	9375	9544	9714	9887	**0061**	**0237**	**0415**	**0595**	29	58	87	116	144
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	97	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	203
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4·3315	3662	4015	4373	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4·7046	7453	7867	8288	8716	9152	9594	**0045**	**0504**	**0970**	73	146	220	293	366
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	175	263	350	438
80	5·671	5·730	5·789	5·850	5·912	5·976	6·041	6·107	6·174	6·243					
81	6·314	6·386	6·460	6·535	6·612	6·691	6·772	6·855	6·940	7·026					
82	7·115	7·207	7·300	7·396	7·495	7·596	7·700	7·806	7·916	8·028					
83	8·144	8·264	8·386	8·513	8·643	8·777	8·915	9·058	9·205	9·357	Differences untrustworthy here.				
84	9·51	9·68	9·84	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

The black type indicates that the integer changes.

TABLE OF CONSTANTS

(Common abbreviations are given in brackets)

Length.

12 inches (in.)	=	1 foot (ft.)
3 feet	=	1 yard (yd.)
1760 yards	=	1 mile (ml. or m.)

10 millimetres (mm.)	=	1 centimetre (cm.)
100 centimetres	=	1 metre (m.)
1000 metres	=	1 kilometre (Km.) or (km.)

Also

1 furlong	=	220 yd	=	$\frac{1}{8}$ ml.
100 links	=	1 chain	=	22 yd.
1 pole *or* rod	=	$5\frac{1}{2}$ yd	=	$\frac{1}{4}$ chain
1 sea-mile	=	10 cables	=	6080 ft.

10 decimetres (dm.)	=	1 metre
1 decametre (Dm.)	=	10 metres
1 hectometre (Hm.)	=	100 metres

Equivalents

1 metre	=	39·37 in.
	=	3·281 ft.
1 Km	=	0·6214 ml (roughly $\frac{5}{8}$ ml.)

1 in.	=	2·540 cm
1 yd.	=	0·9144 m.
1 ml.	=	1·609 Km.

Area.

144 sq. in.	=	1 sq. ft.
9 sq. ft.	=	1 sq. yd.
4840 sq. yd.	=	1 acre (ac.)
640 acres	=	1 sq. ml.

100 sq. cm.	=	1 sq. dm.
100 sq. dm.	=	1 sq. m.
10,000 sq. m.	=	1 sq. Hm.
		or 1 hectare

Also

10 sq. chains	=	1 acre
1 sq. pole *or* rod	=	$30\frac{1}{4}$ sq. yd.
40 sq. poles	=	1 rood
	=	$\frac{1}{4}$ acre

1 are	=	100 sq. m.
100 hectares	=	1 sq. Km.

Equivalents.

1 sq. m.	=	1·196 sq. yd.
1 sq. Km	=	0·3861 sq. ml.
1 hectare	=	2·472 acres

1 sq. in.	=	6·452 sq. cm.
1 sq. yd.	=	0·8361 sq. m.
1 sq. ml.	=	2·590 sq. Km.

Volume.

1728 cub. in. = 1 cub. ft.
27 cub. ft. = 1 cub. yd.
2 pints = 1 quart
4 quarts = 1 gallon

1000 cub. centimetres (c.c.) = 1 cub. dm.
= 1 litre

1 litre = 61·02 cub. in.
= 0·220 gallons
1 cub. in. = 16·39 c.c.
1 gallon = 4·546 litres
= 0·1606 cub. ft.

Weight.

16 ounces (oz.) = 1 pound (lb.)
112 pounds = 1 hundredweight (cwt.)
20 cwt. = 1 ton
= 2240 lb.

1000 grams (grm.) = 1 kilogram (kg.)
1000 kilograms = 1 metric tonne

1 kg. = 2·205 lb.
1 lb. = 0·4536 kg. = 453·6 grm.
1 tonne = 0·984 ton

Foreign Money.

France	Franc of 100 centimes	*Northern America*	Dollar of 100 cents
Germany	Mark of 100 pfennig	*India*	Rupee of 16 annas
Italy	Lira of 100 centesimi	*Ceylon*	Rupee of 100 cents

Useful Constants.

$\pi = 3{\cdot}1416$ 1 radian $= 57{\cdot}296°$ $\varepsilon = 2{\cdot}7183$

$\log_{10}\pi = {\cdot}4971$ $\log_{10}\frac{180}{\pi} = \log_{10} 57{\cdot}296 = 1{\cdot}7581$ $\log_{10}\varepsilon = {\cdot}4343$

$\log_{\varepsilon} 10 = 2{\cdot}3026$

EXAMPLES VIII—(*contd.*)

13. $x = 1, y = 1$ or $x = -\frac{11}{19}, y = -\frac{26}{19}$.
14. $u = -11{\cdot}5, a = -11{\cdot}6$; $s = 5{\cdot}8t^2 - 11{\cdot}5t$.
15. (i) 9, 12. (ii) 10, 2.
16. $\frac{3}{2}$; $\frac{15}{2}$.
17. $A = 0, B = \frac{1}{6}, C = \frac{1}{2}, D = \frac{1}{3}$; 30.
18. 3, 2; 5.
19. (i) 5. (ii) 1, 0.
20. (*a*) 2 2; (*b*) $-3, -4$.
21. 3, 1.
22. $\frac{1}{4}, \frac{1}{3}; +\frac{1}{7}$.
23. (*a*) $1{\cdot}5, -\dot{2}{\cdot}1$; (*b*) -3000, 20,000.
24. (*a*) $(x-2)(2x-3)$; 2, $1\frac{1}{2}$; (*b*) $1\frac{1}{4}$, $2\frac{1}{3}$.
25. (*a*) $(x+4)(2x-1)$; 5, $\frac{1}{2}$, -4; (*b*) 4 in.
26. 2, $\frac{1}{3}$.
27. $6{\cdot}58$, $-3{\cdot}08$.
28. (*a*) $2{\cdot}72$, $\cdot 61$; (*b*) 4, -2.
29. 16, 16 or 1, -4.
30. 5s. 6d.; 8s. 9d.
31. (*a*) 5 or $-\frac{32}{7}$; (*b*) 2 86 or 0 098.
32. (*a*) $\frac{2}{3}$ or $\frac{5}{2}$; (*b*) 1, 2 or $\frac{1}{3}$.
33. (*a*) $\frac{1}{2}$; (*b*) $x = 2\frac{1}{2}, y = 1$.
34. (*a*) $\frac{2MD}{(D^2 - L^2)^2}$; $\frac{2M}{D^3}$. (*b*) 18 97.

CHAPTER II

EXAMPLES IX

1. $\frac{5}{3}$; 4; $\frac{5}{4}$; $\frac{4}{3}$.
2. $\frac{13}{12}$; $\frac{13}{5}$; $\frac{12}{5}$.

EXAMPLES X

1. $\cdot 0049$; $\cdot 9932$; $\cdot 2867$; $\cdot 0349$.
2. 53° 8′; 67° 23′; 36° 52′; 60°.
3. $\theta = 36°\ 52'$.
4. $\theta = 30°$.
5. $\theta = 60°$.
6. $\theta = 45°$.
7. $\theta = 53°\ 8'$ or 0°.
8. $\theta = 60°$.
9. 19° 28′ or 30°.
10. 60°.
11. 14° 29′ or 30°.
12. 30° or 41° 49′.
13. 26° 34′ or 45°.
14. 18° 26′ or 36° 52′.
15. 63° 26′ or 71° 34′.
16. 45° or 71° 34′.
17. 30° or 90°.
18. 53° 8′.

EXAMPLES XI

1. $+0{\cdot}5736$; $-0{\cdot}7660$; $-0{\cdot}9397$; $-0{\cdot}5736$.
2. $-0{\cdot}9397$; $-0{\cdot}3420$; $+0{\cdot}7660$; $+0{\cdot}9781$; $-0{\cdot}6018$.
3. -1; $+5{\cdot}671$; $-0{\cdot}4663$; $-0{\cdot}3640$; $-2{\cdot}7475$.
4. secants: $-1{\cdot}035$; $+1{\cdot}064$; $+2{\cdot}923$; $+2{\cdot}458$.
   cosecants: $-3{\cdot}863$; $-2{\cdot}923$; $+1{\cdot}064$; $-1{\cdot}094$.
5. 30°; 150°; 390°; 510°.
6. 1; $\frac{1}{2}$; $-0{\cdot}7265$.
7. $+1{\cdot}414$; $+1{\cdot}155$; ∞.
8. $-\frac{5}{3}$; $+\frac{4}{3}$.
9. 3rd; 2nd; 3rd; 2nd.
10. $(90 + A)$: $+\cos A$; $-\sin A$; $-\cot A$.
    $(270 - A)$: $-\cos A$; $-\sin A$; $+\cot A$.
11. Sin A, $-\cos A$, $-\tan A$; $-\sin A$, $\cos A$, $-\tan A$.
12. Sec A, $-\operatorname{cosec} A$, $-\tan A$; $-\sec A$, $\operatorname{cosec} A$, $-\tan A$.

EXAMPLES XII

3. N. 78° 41′ W.; 3·61 miles. 4. 1358 yd.; 1576 yd. 5. 3·30 m.
6. 7·77 m. 7. 7° 28′ or 332° 32′. 8. 105°.
9. 160°. 10. 174° 2′. 11. 5·51 m., 1°.

EXAMPLES XIII

1. $\hat{C} = 95°$; $a = 3·34$ in.; $c = 4·34$ in.
2. $\hat{A} = 58° 29'$ *or* $121° 31'$; $\hat{B} = 78° 31'$ *or* $15° 20'$; $b = 14·37$ *ft. or* 3·915 ft.
3. $\hat{C} = 49°$; $a = 2·36$ miles; $b = 2·66$ miles.
4. $CA = 734$ yd.; $CB = 898$ yd.
5. 312 yd. 6. 55·1 yd. 7. 3·97 miles.
8. 28·3 ft. 9. 22·7 ft. 10. 1·18 m.
11. 67·8 ft.

EXAMPLES XIV

3. $\frac{56}{65}, \frac{63}{65}$. 4. $\frac{304}{425}, -\frac{87}{425}$. 5. $\frac{336}{625}, \frac{527}{625}$.
6. 4·25, 0·4375. 7. 1. 8. $1/\sqrt{3}$.
9. $\frac{1}{2}$. 10. $1/\sqrt{2}$.

EXAMPLES XV

2. $\theta = 36° 52'$.
3. *53° 8′; 36° 52′; 50° 12′; 2; 0·2655; 4.*
4. $-0·866$; $-0·5$; $+1·7321$.
5. $+\cos A$; $-\cos A$; $+\cot A$; $-\sec A$; $+\sec A$.
6. $+0·5$; $-1·0$; -2; $+1·414$; 0.
7. $A\hat{B}C = 95°$; $a = 6·68$, $b = 8·68$.
8. 7·21 miles; 3·98 miles. 9. 874 yd.
10. $4\cos^3 A - 3\cos A$; $\dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.
11. *(c) 6107·9; 6045·7.* 12. 2·164 miles; 2·677 miles.
13. (*a*) 0; (*b*) 3·261. 14. (*a*) 3; (*b*) $\dfrac{r^2\theta}{2}$.
15. (c) 60° or 75° 31′. 16. 3·64 miles; N.W.
17. $A\hat{B}C = 97°$; $A\hat{C}B = 39°$; $AB = 3·17$ in.
18. 0·365 mm.
19. (*a*) 2·793; 2·967; 3·142; 3·317; $-0·3640$; $-0·1763$; 0; $+0·1763$; (*b*) 53·37.
20. *14·6 miles; 2.40* p.m. 21. *(a) 19·7* sq. in.; *(b) Ellipse, 14·5* sq. *in.*

CHAPTER III

EXAMPLES XVI

1. 7·55 lb. wt.; 66° 34′. 2. (i) 12·61 oz.; (ii) 11·93 lb.; (iii) 5 tons.
3. In a direction making an angle of 138° 36′ with that of the stream; 2 min. 35 sec.
4. 1 min. 42 sec.; 150 yd.

EXAMPLES XVII

1. 7·55 units.
2. 36·06 lb. making angles of 146° 18′ with the downward force and 123° 42′ with the horizontal force.
3. (i) 9·98 lb.; (ii) 56° 15′; (iii) 4 tons; (iv) 158·8 grm.
4. N. 3° 28′ W.; 12·42 m.p.h. or N. 76° 32′ W.; 2·90 m.p.h.
5. 3·38 ft. per sec. per sec. in a direction making an angle of 115° with the initial velocity.
6. 24 m.p.h.; N. 18° 46′ W.

EXAMPLES XVIII

1. 21·7 m.p.h. in direction S. 71° 53′ W.
2. N. 8° 54′ E.
3. 64° 32′.
4. 395 m.p.h., N. 19° 20′ E.
5. S. 81° 52′ E.
6. N. 45° 35′ W., 46 min.

EXAMPLES XIX

1. 1·05 tons making an angle of 84° 12′ with the 2 ton force.
2. 7·23 lb.; N. 4° 2′ W.
3. No. 29·2 lb., S. 10° 42′ E.

EXAMPLES XX

1. 73·8 lb.; 30·8 lb.
2. 36·1 ft. per sec.
3. 116·9 lb.; 33·8 lb.
4. 12·15 lb.; N. 48° 51′ E.
5. 25 lb., S. 42½° E.

EXAMPLES XXI

1. 11·62 tons in a direction making an angle of 20° 39′ with the 8 lb. force.
2. 146·5 lb.; 136 lb.
3. In a direction making an angle of 120° with the current; 1 min. 11 sec.
4. 89·9 m.p.h. in a direction making an angle of 27° 5′ with the horizontal.
5. 12·75 lb.; N. 25° 15′ W. or S. 25° 15′ E.
6. 9·4 m.p.h. per sec.; S. 4° 59′ E.
7. 38·4 lb.
8. 4·36 tons in a direction making an angle of 23° 24′ with the 3 ton force.
9. 64·8 ft. per sec.; S. 64° 7′ E., 324 ft.
10. 40·9 sec.
11. 20·62 lb.; S. 73° 3′ E.
12. 3·14 tons; 0·116 tons.
13. 4·375 lb.; 7·93 lb.
14. 22·26 lb.; N. 27° 51′ W.
16. 6478 units; 2150 units per sec.
17. 5·87 tons; S. 61° 22′ W.
19. 8·32 tons; 24°.
20. 1782 ft. per sec.; 1774 ft. per sec.
21. 61 lb.
22. 29½ min., 17·8 miles.
24. 10·2 units, N. 59° E.

CHAPTER IV

EXAMPLES XXII

1. 38 sq. ft.
2. 6·28 sq. in.
3. 16·49 ft.
4. 3 ft. 10 in.
5. 2·905 in.
6. 12·28 ft.
7. 9·1 sq. cm.
8. 0·645 sq. in.
10. 0·414 in.

EXAMPLES XXIII

1. 21° 29′.
2. 8 cm.
3. 240°; 200°.
4. 70°.
6. 3·78 ft.; 32 ft.
7. 56·2 in.
8. 3 in., 43·85 in.
9. 85·3 in.
10. 37·6 in.
11. 4·24 in.
12. 5·11 in.

EXAMPLES XXIV

2. $\frac{4}{9}A$; $\frac{5}{9}A$.
3. 16 ft.; 8 ft.; 32 ft.
4. 10·04 sq. in.
5. 72·39 sq. in.
6. 9·77 miles.
7. 4 ft. 1 in.
8. 18 in.
9. 80 sq. yd.
10. 11,406 sq. ft.
11. $\frac{1}{125,000}$.
12. $1\frac{131}{169}$ in., $4\frac{44}{169}$ in.

EXAMPLES XXV

1. 34·44 lb.; 432·7 sq. in.
2. 14·58 cub. in.; 186·5 cub. in.
3. 34·9 sq. in., 11 cub. in.
4. 248 cub. in.
5. 7·18 c.c.
6. 14,727 lb.
7. 820 cub. ft.; 26·16 ft.
8. 45·45 ft.; 83,980 cub. ft.
9. 4·81 sq. in., 0·693 cub. in.
10. 2720 sq. cm., 7240 c.c.
11. $8\frac{1}{2}$ in., 54·2 sq. in., 22·9 cub. in.

EXAMPLES XXVI

1. 5984 sq. cm.; 9792 cub. cm.
2. 4374 sq. cm.; 7289 cub. cm.
3. 1060; 0·5.
4. 15·9 cub. in.
5. 72 yd.; 2·08 ac.
6. 37·3 ft. per sec.; 597 ft.

EXAMPLES XXVII

1. 20·4 ft.
2. 2 ft. $4\frac{1}{2}$ in.
3. 6·81 in.
4. 103·6 sq. in.
5. 36° 52′.
6. $P\hat{A}B = 70°$; $P\hat{B}A = 60°$.
7. 6·4 in.
9. $10\frac{1}{4}$ per cent.
10. $\frac{0·815}{1}$.
11. 1568 cub. in.; 896 sq. in.; 1372 cub. in.
12. $14\frac{1}{2}$ ft.
13. 71·1 acres; $2\frac{1}{4}$ in.
14. 0·0818 cub. in.
15. 10·4 sq. in.
16. 3535 cub. in.
17. Vol. $= 3\pi L(D^2 - d^2)$ cub. in.; $12\pi L(D + d)$ sq. in. Volume reduced to $\frac{1}{4}$; surface halved.
18. $\Sigma v \,.\, \delta t = 3880$
 = Distance (in feet) travelled by train in 90 sec.
19. $\frac{1}{2}r^2\theta$; 17·22 sq. ft.
20. 20·31 ft.; 55·26 sq. ft.
21. 0·3628 sq. in.

CHAPTER V

EXAMPLES XXVIII

1. $y = 2x + 1$, $y = x + 7$, $y + 3x - 2 = 0$, $3y = x - 1$.
2. $2, \frac{1}{2}, -3, -\frac{2}{5}$.
3. $2, \frac{3}{8}, 2, \frac{4}{3}$; $y = 2x - 10$, $8y = 3x + 41$, $y = 2x$, $3y = 4x - 24$.
4. $-\frac{4}{3}, -\frac{1}{2}, -\frac{1}{2}, -4$; $3y + 4x - 29 = 0$, $2y + x - 9 = 0$, $2y + x = 0$, $y + 4x - 8 = 0$.

EXAMPLES XXVIII—(*contd.*)

5. 5, 13, 25.
6. $x^2 + y^2 - 6x + 4y - 156 = 0$.
7. (i) -2, 3; 4. (ii) $\frac{3}{2}$, $\frac{5}{4}$; $\frac{13}{4}$.
8. $x^2 + y^2 - 8x - 6y - 24 = 0$. 9. 5, 3. 10. 6, 4.
11. $x^2 + y^2 + 14x + 14y + 49 = 0$.
12. $x^2 + y^2 - 10x - 10y + 25 = 0$, $x^2 + y^2 - 26x - 26y + 169 = 0$.

EXAMPLES XXIX

5. 2·19 or $-$·69. 6. $x = -1{\cdot}21$, $y = -{\cdot}42$ or $x = {\cdot}55$, $y = 3{\cdot}1$.
7. $x = {\cdot}133$, $y = 4{\cdot}43$ or $x = -1{\cdot}88$ $y = 5\,44$.
8. $x = 3{\cdot}31$, $y = 0\,65$; or $x = -1{\cdot}81$, $y = -1{\cdot}91$.
9. $x = 1{\cdot}23$, $y = -0{\cdot}11$; or $x = -0{\cdot}10$, $y = 0{\cdot}56$.
10. $x = -0{\cdot}44$, $y = 0{\cdot}56$; or $x = -4{\cdot}56$, $y = -3{\cdot}56$.

EXAMPLES XXXI

1. 12 2, $-$4·9, 2·88, $-$2·57.

EXAMPLES XXXII

1. ·83, 1·88, 1·78, ·32, 25·7.
2. 3·6, 280, 2 49.
3. 1 35, $-$0 31.
7. $16y = 3x^2 - 6x + 3$.
8. $x^2 + y^2 = 25$.
9. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
17. $r = 2\cos\left(\theta - \frac{\pi}{6}\right)$.
18. $2r \sin \theta = 3$.

EXAMPLES XXXIII

1. 1·24, $-$4·57; $3x^2 + 10x - 17 = 0$.
2. 3 90 or ·77.
3. 1·69 or $-$·44.
4. 850 tons.
5. 1·9, $-$·4, $-$1·5
6. 1·55, $-$·21.
7. 1·1.
8. $y = 6{\cdot}75x$.
9. $-$2 1, 1·3, 2·8.
10. ·55, 1·65, 2·8.
11. $x = 1$, $y = -5$ or $x = -5$, $y = -1$.
12. $x = 2{\cdot}6$, $-{\cdot}96$; $x^4 + 9x^3 - 27x - 36 =$
    $y = 2{\cdot}2$, ·32; $(y^3 + 3y - 4)^3 = 27y$.
13. 63. 14. 1·4. 15. 2·5, $-$3·5.
16. 1, $-$1·5. 17. 2·3, ·6, $-$2·9. 18. ·48; £624.
19. 2·62, ·38, $-$1. 20. 7·37.
21. (i) 18·2. (ii) 2·47 (iii) 6·1. 22. (i) 5·2, 2·5, $-$1·7. (ii) 6·5, 1·9 $-$ 1·4.
23. 1·02, $-$1·37. 24. $x = 1{\cdot}49$, $t = 4{\cdot}04$.
25. $x = {\cdot}023a$, $y = {\cdot}133a$; $x = 1{\cdot}649a$; $y = 1{\cdot}707a$.
26. 41¾°, 318¼°. 28. $r = 0\,005A$, 200°.

CHAPTER VI

EXAMPLES XXXIV

1. $\frac{4}{3}$. 2. 12. 3. 13. 4. -1.
5. $\frac{1}{2}, -\frac{1}{4}$. 6. 2, 4, or -2, -4. 7. $\pm\sqrt{2}$, 0.

EXAMPLES XXXV

1. $7x^6$. 2. $12x^3$. 3. $25x^4$. 4. $91x^{12}$.
5. $-9x^2$. 6. $2 + 6x$. 7. 4. 8. $3x^2 - 2$.
9. $4x^3 - 6x$. 10. $56x^6 - 56x^7$. 11. $12x^{11} - 12$.
12. $4 + 8x$. 13. $4x^3 + 6x^2 + 2x$. 14. $-3 + 6x - 3x^2$.
15. $-24x + 24x^3 - 6x^5$. 16. $2x - 2$.
17. (i) 6 f.s., -12 f.s.s.; (ii) -30 f.s., -24 f.s.s.
18. (i) 12 yd. per min. East; (ii) 3 min.; (iii) 27 yd.
(iv) 102 yd. per min. West; (v) $s = 1200 - 102t$.
19. 0·0132 sq. in. 20. 0·282 sq. in. per sec. 21. 186·8.
22. 0·02. 23. $\sqrt{3}ab/2$. 24. $\frac{V}{\pi}\left(\frac{h}{ax}\right)^2$ in. per sec.

EXAMPLES XXXVI

1. $y = 8x - 8$. 2. $y = 8x - 5$. 3. $y = 10x - 16$.
4. $9y - 9x = 2$, $9y - 9x + 2 = 0$. 5. 1, -8.
6. 3 and -1. 7. -1 (maximum), 3 (minimum).
8. Maximum $= 4\frac{5}{27}$, minimum $= 3$.
9. Minimum $\frac{5}{16}$, no maximum.
10. 1, inflexion; -2, minimum.

EXAMPLES XXXVII

1. $3\cos(3x + 4)$. 2. $-5\sin 5x$. 3. $-6\cos(4 - 2x)$.
4. $7\sin(5 - 7x)$. 5. $-\frac{3\pi}{180}\sin 3x^\circ$. 6. $12\cos\left(3x - \frac{\pi}{4}\right)$.
7. $-6\sin(3x - 7)$. 8. $\frac{\pi}{30}\cos(6x - 20)^\circ$. 9. $-4\sin\left(2x - \frac{\pi}{6}\right)$.
10. $2\sin\left(\frac{\pi}{3} - 2x\right)$. 11. $-2\sin 2x + 2\cos 2x$.
12. 0·464 rad. 13. 1·107 rad. 14. 1·893 rad.
15. $\frac{2\pi}{3}$. 16. $\frac{\pi}{4}$.

EXAMPLES XXXVIII

1. 8·0023 c.c. 2. $\frac{7}{9}$, 1; $\frac{8}{9}$.
3. $\frac{\pi b a^2 x^2}{h^2}$ cub. in. per sec. 4. $y = 3x - 9$.
5. 1. 6. $1\frac{1}{3}$, 2; $1\frac{2}{3}$.
7. $3x^2 - 6x + 5 = 0$ has no real roots; at $x = 1$.
8. (a) 200, 72 ft. per sec.; $6\frac{1}{4}$ sec.; (b) 3.
9. $a = \frac{3}{16}$, $b = \frac{43}{16}$; -2, 1, 4. 10. $a = \frac{5}{4}$, $b = -4$, $c = 9·6$.

EXAMPLES XXXVIII—(*contd.*)

11. $a = \frac{36}{23}, b = -\frac{15}{23}, c = \frac{2}{23}.$

12. Side of base = 6 ft.,
Height = 3 ft.

ADDITIONAL ELEMENTARY EXAMPLES

1. (i) $x = \frac{1}{3}$; (ii) $x = -1, y = -2, z = 1.$

2. (i) 1·29 or − 3·29; (ii) − 0·192 or − 7·81.

3. (i) $x = \frac{3}{2}, y = 6, x = -\frac{3}{2}, y = -6;$

   (ii) $x = \frac{1}{a}, y = \frac{1}{b}, x = -\frac{1}{a}, y = -\frac{1}{b}.$

4. (i) $x = 1{\cdot}26, y = 2{\cdot}58.$

   (ii) $x = 1, y = 1$, or $x = \frac{3}{2}, y = \frac{1}{4}.$

5. 25 tons, 15 tons; £1 17s., £2 9s. 6. 1·97 sec.

7. 11 in. 11. 48° 11′ or 180°.

12. 2·37 miles, N. 83° 42′ E. 13. $A = 112°, b = 2{\cdot}532, c = 3{\cdot}465.$

14. $B = 61° 3', \quad C = 88° 57', c = 7{\cdot}938$; or
$B = 118° 57', C = 31° 3', c = 4{\cdot}126.$

15. $\cos A = \pm \frac{5}{13}, \sin 2A = \pm \frac{120}{169}, \cos 2A = -\frac{119}{169}.$

16. $\sin (X + Y) = \frac{56}{65}, \tan (X - Y) = \frac{16}{63}.$

17. $-\cos A, \sin A, \tan A.$ 19. 9·04 lb., 138° 4′ E. of N.

21. 20·56, N. 6° 36′ W. 22. 31·34 ft. per sec., N. 22° 15′ W.

23. 136° 24′ with direction of current. 24. (i) 1·847 in.; (ii) 1·942 in.

25. 5 in., 31·42 sq. in., 14·66 cub. in.

26. 211 sq. ft.

27. 73° 44′, 37·7 cub. in., 47·12 sq. in.; 216°.

28. 85·3 in. 29. − 0·56, 2, 3·56.

30. − 2·07, 1·17, 2·9. 31. 1·1.

32. $4y = 27x$; $4x^3 + 12x^2 - 15x + 4 = 0.$

33. $2, -\frac{2}{3}.$ 34. $-\frac{1}{2}$

35. $y = (4x_1 - 3x_1^2)x - 2x_1^3 + 2x_1^2$; $y = x$

36. − 2; 4; 1.

37. $9x^2 - 6x + 2 = 0$ has no real roots; at $x = \frac{1}{3}.$

38. $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at $x = \frac{1}{3}.$

39. $-2 \cos (4 - 2x), 2 \sin (4 - 2x), \frac{\pi}{36} \cos (120 + 5x)°$